

Conformal collider physics: energy and charge correlations

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ABSTRACT: We study observables in a conformal field theory which are very closely related to the ones used to describe hadronic events at colliders. We focus on the correlation functions of the energies deposited on calorimeters placed at a large distance from the collision. We consider initial states produced by an operator insertion and we study some general properties of the energy correlation functions for conformal field theories. We argue that the small angle singularities of energy correlation functions are controlled by the twist of non-local light-ray operators with a definite spin. We relate the charge two point function to a particular moment of the parton distribution functions appearing in deep inelastic scattering. The one point energy correlation functions are characterized by a few numbers. For $\mathcal{N} = 1$ superconformal theories the one point function for states created by the R-current or the stress tensor are determined by the two parameters a and c characterizing the conformal anomaly. Demanding that the measured energies are positive we get bounds on a/c . We also give a prescription for computing the energy and charge correlation functions in theories that have a gravity dual. The prescription amounts to probing the falling string state as it crosses the AdS horizon with gravitational shock waves. In the leading, two derivative, gravity approximation the energy is uniformly distributed on the sphere at infinity, with no fluctuations. We compute the stringy corrections and we show that they lead to small, non-gaussian, fluctuations in the energy distribution. Corrections to the one point functions or antenna patterns are related to higher derivative corrections in the bulk.

KEYWORDS: Strong Coupling Expansion, AdS-CFT Correspondence, Jets.

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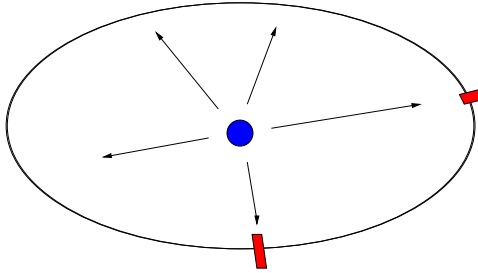


Figure 1: A localized excitation is produced in a conformal field theory and its decay products are measured by calorimeters sitting far away.

1. Introduction

In this paper we consider conformal field theories and we study physical processes that are closely related to the ones studied at particle colliders. In some sense we will be studying “conformal collider physics”. We consider an external perturbation that is localized in space and time near $t \sim \vec{x} \sim 0$. This external perturbation couples to some operator \mathcal{O} of the conformal field theory and produces a localized excitation in the conformal field theory. This excitation then grows in size and propagates outwards. We want to study the properties of the state that is produced. For this purpose we consider idealized “calorimeters” that measure the total flux of energy per unit angle far away from the region where the localized perturbation was concentrated. As a particular example one could have in mind a real world process $e^+e^- \rightarrow \gamma^* \rightarrow \text{hadrons}$,¹ where we produce hadrons via an intermediate off shell photon. We can treat the process to lowest order in the electromagnetic coupling constant and to all orders in the strong coupling constant. The QCD computation reduces to studying the state created on the QCD vacuum by the electromagnetic current j_{em}^μ . From the point of view of QCD this current is simply a global symmetry. In this case the theory is not conformal, but at high enough energies we can approximate the process as a conformal one to the extent that we can ignore the running of the coupling and the details of the hadronization process. In this paper we will analyze similar processes but in conformal field theories.

Our goal is to describe features of the produced state. For example, at weak coupling we expect to see a certain number of fairly well defined jets. At strong coupling we expect to see a more spherically symmetric distribution [2–4]. We need suitably inclusive variables which are IR finite. In QCD this is commonly done using inclusive jet observables [5], see [6] for a review. In this paper we study a particularly simple set of inclusive observables which are the energy correlation functions, originally introduced in [7]. They are defined as follows. We place calorimeters at angles $\theta_1, \dots, \theta_n$ and we measure the total energy per unit angle deposited at each of these angles. We multiply all these energies together and compute the average over all events. These are also inclusive, IR finite observables which

¹For early work on the applications of scale invariance to strong interactions and, in particular, e^+e^- collisions, see [1].

one could use to study properties of the produced state. Energy correlation functions for hadronic final states have been measured experimentally and they are one of the ways of making precise determinations of α_s (see [8] for example).

A nice feature of energy correlation functions is that they are defined in terms of correlation functions of local gauge invariant operators. They are given in terms of the stress tensor operator [9]. More precisely, consider the expression for the integrated energy flux per unit angle at a large sphere of radius r

$$\mathcal{E}(\theta) = \lim_{r \rightarrow \infty} r^2 \int_{-\infty}^{\infty} dt n^i T_i^0(t, r\vec{n}^i) \quad (1.1)$$

where n^i is a unit vector in R^3 and it specifies the point on the S^2 at infinity where we have our “calorimeter”. If we integrate this quantity over all angles we get the total energy flux which is equal to the energy deposited by the operator insertion. Energy correlation functions are defined as the quantum expectation value of a product of energy flux operators on the state produced by the localized operator insertion

$$\langle \mathcal{E}(\theta_1) \cdots \mathcal{E}(\theta_n) \rangle \equiv \frac{\langle 0 | \mathcal{O}^\dagger \mathcal{E}(\theta_1) \cdots \mathcal{E}(\theta_n) \mathcal{O} | 0 \rangle}{\langle 0 | \mathcal{O}^\dagger \mathcal{O} | 0 \rangle} \quad (1.2)$$

where \mathcal{O} is the operator creating the localized perturbation. Note that the operators are ordered as written, they are not time ordered. Notice, also, that the expectation values in the left hand side of (1.2) are defined on the particular state created by the operator \mathcal{O} and they are *not* vacuum expectation values. The energy operators are very far away from each other and they commute with each other. This will become more clear below when we think of the operators as acting on null outgoing infinity, sometimes called \mathcal{I}^+ . Of course, we usually think of the energy deposited at various calorimeters as commuting observables, since we measure them simultaneously. Notice that when we compute an n point function we place calorimeters at n points but we also allow energy to go through the regions where we have not placed calorimeters.

In this paper we will assume that we have a conformal field theory. There are several motivations for doing so. First, the conformal case is simpler because it has more symmetry and, at the same time, it allows us to consider theories that are strongly coupled. There are some interesting statements that can be made using conformal symmetry. Second, we could have a theory for new physics beyond the Standard Model which is conformal, as in the Randall-Sundrum II [10] or the unparticle [11] scenarios, or approximately conformal, as in the “hidden valley” scenario [12]. One would like to describe the events in these theories. In order for energy correlations to be observable to us we need some way to transfer the energy from the new sector back to the standard model, as in [12]. Depending on the details, this conformal breaking and conversion process might or might not destroy the energy correlations one computes in the conformal theory. We will not discuss this problem here. A similar issue arises in QCD. For a sample of references on the influence of hadronization on energy correlations for QCD see [7, 13]. The final motivation is a more theoretical one, which is to understand better the AdS/CFT correspondence [14–16]. Energy correlations are natural observables on the field theory side which one would

like to understand using gravity and string theory in AdS. We will see that on the gravity side, energy correlations translate into the probing of a string state, created by the localized perturbation, with a gravitational shock wave as it falls into the AdS horizon. Thus, the problem becomes a high energy scattering calculation in the bulk.

This paper is organized as follows. In section two we make some general remarks on energy correlation functions in conformal field theories. By making conformal transformations we can picture the problem in various ways. We also make some remarks on the small angle behavior of the correlators when two of the energy operators come close together. We point out that this small angle behavior can be analyzed by means of an operator product expansion which involves non-local light-ray operators which are closely related to the ones that appear in the discussion of deep inelastic scattering. We also relate a moment of the deep inelastic cross section, or parton distribution function, to a particular energy two point correlation function. Finally, we consider the general form of the energy one point function $\langle \mathcal{E}(\theta) \rangle$ and relate it to vacuum expectation values of three point functions.

In section three we study conformal field theories that have a gravity (or string theory) dual and we describe a prescription for computing the energy correlation functions. The procedure amounts to taking a “snapshot” of the wavefunction of the state produced by the operator insertion. In the gravity approximation we find that the energy correlation functions are perfectly spherically symmetric as was expected from the very rapid fragmentation that one expects at strong coupling. This phenomenon was originally analyzed in deep inelastic processes in [2] (see also [4]).

In section four we discuss the leading stringy corrections. They amount to small fluctuations in the energy distribution of order $1/\sqrt{\lambda}$. We also consider these corrections for charge correlations which have interesting features in the case that the charges are carried by flavor symmetries. Finally, we study the regime where two of the angles come close together and find that the result is determined by the energy of peculiar non-local string states which are dual to the light-ray operators that appeared in the general field theory discussion. These operators have a high conformal dimension at strong coupling going like $\Delta \sim \lambda^{1/4}$.

In section five we present a summary, conclusions and a discussion of open problems.

2. Energy correlations in conformal field theories

In this section we study energy correlation functions in general conformal field theories. The discussion in this section is valid for any value of the coupling.

2.1 Energy correlations in various coordinates systems

The goal of this subsection is to think about energy correlations in various coordinate systems in order to make manifest its various properties and also in order to simplify later computations.

It is interesting to take a step back and think about the energy density as follows. For any generator, G , of the conformal group there is an associated conformal killing vector ζ_G^μ ($x^\mu \rightarrow x^\mu + \zeta_G^\mu$). The associated conserved charge can be written as the integral of

a conserved current, constructed by contracting ζ_G^μ with the stress tensor, over a spatial hypersurface

$$Q_G = \int_{\Sigma_3} *_{4} j_G, \quad j_G^\mu \equiv T_{\mu\nu} \zeta_G^\nu \quad (2.1)$$

where the normalization of the stress tensor is chosen so that $T_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}$. This expression of the charges is covariant under conformal transformations. It is also invariant under Weyl transformations of the four dimensional metric² $g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}$, $T_{\mu\nu} \rightarrow \Omega^{-2} T_{\mu\nu}$.

It is convenient to understand clearly the symmetries of the problem. We are interested in measuring the flux of energy at large distances. Thus, we focus our attention on the boundary of Minkowski space $R^{1,3}$. The conformal generators that leave the boundary fixed are the dilatation and the Poincare generators, including the translations P^μ and the $SO(1,3)$ lorentz transformations. In other words, we have the whole conformal group except the special conformal transformations. In order to see that the large r limit in (1.1) is well defined, and also to gain some more insight into the problem, it is convenient to perform a conformal transformation from the original coordinates x^μ to new coordinates y^μ . The new coordinates are such that the future boundary of the original Minkowski space is mapped to the null surface $y^+ = 0$. The explicit change of coordinates is³

$$y^+ = -\frac{1}{x^+}, \quad y^- = x^- - \frac{x_1^2 + x_2^2}{x^+}, \quad y^1 = \frac{x^1}{x^+}, \quad y^2 = \frac{x^2}{x^+} \quad (2.2)$$

where $y^\pm = y^0 \pm y^3$, and similarly for x^\pm . The inverse change of coordinates is given by the same expressions with $x \leftrightarrow y$. The advantage of the new coordinates is that now the energy is expressed in terms of an integral over the surface at $y^+ = 0$ and we do not have to take any limit, such as the large r limit in (1.1). Actually, to be more precise, the surface $y^+ = 0$ corresponds to the future lightlike boundary of Minkowski space. The energy correlation function (1.1) involves an integral over the past and the future boundaries of Minkowski space. However, in the physical situation we are interested in, where we have the vacuum in the past, there is no contribution from the past light-like boundary and we can focus only on the future boundary. Of course, one could also directly define the energy flux operator in terms of an integral over only the future boundary.

In order to switch between different coordinate systems it is convenient to think about $R^{1,3}$ as follows. We introduce the six coordinates Z^M subject to the identification $Z^M \sim \lambda Z^M$ and the constraint⁴

$$-(Z^{-1})^2 - (Z^0)^2 + (Z^1)^2 + (Z^2)^2 + (Z^3)^2 + (Z^4)^2 = 0 \quad (2.3)$$

The usual coordinates on $R^{1,3}$ are projective coordinates $x^\mu = \frac{Z^\mu}{Z^{-1} + Z^4}$, $\mu = 0, 1, 2, 3$. The metric induced on this surface, (2.3), by the $R^{2,4}$ metric is fixed up to an overall x -dependent factor. We can choose a metric by choosing a ‘‘gauge condition’’ such as

²We are ignoring the conformal anomaly since it only contributes as a c -number, independent of the quantum state of the field theory.

³This type of coordinates have also been studied in [80].

⁴Note that Z^{-1} is the ‘‘minus one’’ component of the vector Z and it does not denote the inverse of Z . Hopefully, this notation will not cause confusion.

$Z^{-1} + Z^4 = 1$. Different “gauge conditions” lead to metrics that differ by a Weyl rescaling. The coordinates y^μ in (2.2) correspond to the choice

$$y^0 = -\frac{Z^{-1}}{Z^0 + Z^3}, \quad y^3 = -\frac{Z^4}{Z^0 + Z^3}, \quad y^1 = \frac{Z^1}{Z^0 + Z^3}, \quad y^2 = \frac{Z^2}{Z^0 + Z^3} \quad (2.4)$$

In fact, using (2.4) and (2.2) we can easily go between the two sets of coordinates. We have $dx^\mu dx_\mu = \frac{dy^\mu dy_\mu}{(y^+)^2}$. We also clearly see that (2.2) amounts to a $\frac{\pi}{2}$ rotation in the $[-1,0]$ plane and in the $[4,3]$ plane of $R^{2,4}$, which is an element of the conformal group. The boundary of Minkowski space is the null surface given by $Z^{-1} + Z^4 = 0$. We can think of the various generators of the conformal group as the antisymmetric matrices $M^{[MN]}$ which generate the transformations $\delta Z^N = M^{[NM]} Z_M$.⁵ Defining $Z^\pm = Z^{-1} \pm Z^4$, we can see that all the generators that leave the surface $Z^+ = 0$ invariant are all the ones with no + index plus the generator $M^{[+-]}$. In this language the four momentum generators in the x coordinates correspond to $M^{[-\mu]}$, $\mu = 0, 1, 2, 3, 4$. These generators have a particularly simple form at $Z^+ = 0$

$$P_\mu \sim Z_\mu \frac{\partial}{\partial Z^-} - Z_- \frac{\partial}{\partial Z^\mu} \quad \longrightarrow \quad P_\mu|_{Z^+=0} \sim Z_\mu \frac{\partial}{\partial Z^-} \quad (2.5)$$

(note that $Z_- = -Z^+/2$). Since the Killing vectors are all proportional to each other, then all four generators involve a single component of the stress energy tensor. Using (2.1), (2.4) and (2.5) we can write

$$\begin{aligned} P_x^0 + P_x^3 &= \int dy_1 dy_2 \mathcal{E}(y_1, y_2) \\ P_x^0 - P_x^3 &= \int dy_1 dy_2 (y_1^2 + y_2^2) \mathcal{E}(y_1, y_2) \\ P_x^1 &= \int dy_1 dy_2 y^1 \mathcal{E}(y_1, y_2) \\ P_x^2 &= \int dy_1 dy_2 y^2 \mathcal{E}(y_1, y_2) \mathcal{E}(y_1, y_2) \equiv 2 \int_{-\infty}^{\infty} dy^- T_{--}(y^-, y^+ = 0, y^1, y^2) \end{aligned} \quad (2.6)$$

We see that they are all determined by T_{--} thanks to the simple form of the generators at $Z^+ = 0$ (2.5). The conclusion is that we are computing correlation functions of T_{--} and these determine all the components of the energy and the momentum. These expressions have the advantage that no limit is involved but they have the disadvantage that the $SO(3)$ rotation symmetry is not manifest. Since no limit is involved, it is clear that the expectation values of (2.6) will be finite. In fact, we are considering an external operator insertion which is localized in x space. This implies, in particular, that it is localized near $x^+ \sim 0$ so that it is far enough from $y^+ = 0$ which is the point where we insert the operators (2.6).

We should note that the dilatation symmetry of the original coordinates $x^\mu \rightarrow \lambda x^\mu$ becomes a boost in the y^+, y^- plane in the y coordinates (2.2). Similarly the dilatation transformation in the y variables becomes a boost in the x^\pm plane.

⁵This $SO(2,4)$ manifestly invariant formalism has also been studied recently in [78].

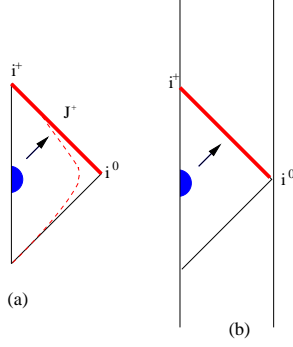


Figure 2: (a) Penrose diagram of flat Minkowski space. The dotted line is a surface at constant r where we measure the energy flux. In the large r limit this becomes the light-like boundary, \mathcal{J}^+ , of Minkowski space. We consider only the future part of the boundary. The semicircle represents a localized operator insertion. In (b) we extend the coordinates to the conformal completion of Minkowski space, which gives us $S^3 \times R$. The future boundary of the original space is simply the light-cone of the point at spatial infinity, i^0 .

An alternative point of view is the following. We write the original coordinates as

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2 = r^2 \left[\frac{-dt^2 + dr^2}{r^2} + d\Omega_2^2 \right] \quad (2.7)$$

The original metric and the bracketed metric in (2.7) differ by a Weyl transformation, but such a transformation leaves the physics of the CFT invariant. So we can view our CFT as defined on an extremal black hole: $AdS^2 \times S^2$. Then, the boundary of Minkowski space corresponds to the black hole horizon situated at $t, r = \infty$. We see that we can view our measurement as one done at the horizon of an extremal black hole. (Of course we can also consider other coordinates related by Weyl transformations which would suggest other pictures.) By introducing new coordinates we can write the AdS_2 metric in (2.7) as

$$ds^2 = \frac{-dt^2 + dr^2}{r^2} = \frac{-d\tau^2 + d\sigma^2}{\sin^2 \sigma}, \quad t = \frac{\sin \tau}{\cos \tau + \cos \sigma}, \quad r = \frac{\sin \sigma}{\cos \tau + \cos \sigma} \quad (2.8)$$

The horizon is at $\tau^+ \equiv \tau + \sigma = \pi$. We also define $\tau^- = \tau - \sigma$. We can then write the generators (2.6) as

$$P^0 = \int d\Omega_2 \mathcal{E}(\vec{n})$$

$$P^i = \int d\Omega_2 n^i \mathcal{E}(\vec{n}) \mathcal{E}(\vec{n}) \equiv 2 \int_{\tau^+=\pi} d\tau^- \left(\cos \frac{\tau^-}{2} \right)^2 T_{\tau^- \tau^-} \quad (2.9)$$

where n^i is a unit vector in R^3 and specifies a point on S^2 . In these coordinates the $SO(3)$ rotation symmetry is manifest. The fact that the energy flux and the momentum flux is related to the same operator, T_{--} , is indeed what we would naively expect in a theory of massless particles. Namely, if at some point of the sphere we have energy $\mathcal{E}(\theta)$ then we have momentum $P^i = n^i \mathcal{E}(\theta)$. Here we have shown that this also holds for a general interacting CFT. This is due to the simple form of the Killing vector (2.5) at $Z^+ = 0$.

Note also that the $SO(1,3)$ Lorentz symmetry acts on the 2-sphere as the $SL(2, C)$ group of conformal transformations of S^2 . Our problem however, does not reduce to computing correlators in a 2d CFT, since the state we are considering breaks the $SL(2, C)$ invariance. Under these transformations the operator \mathcal{E} transforms as a dimension three operator. The easiest way to see this is to recall that these $SL(2, C)$ transformations are the ordinary Lorentz transformations of the original coordinates. In particular we have seen that x^\pm boosts become dilatation operators in the y variables. In those variables it is clear that $\int dy^- T_{--}$ has dimension three. In particular, one can find the relation between the operator $\mathcal{E}(y^1, y^2)$ which is defined on a plane to the one on the sphere, $\mathcal{E}(\vec{n})$, by following the coordinate transformation between the plane and the sphere at $Z^+ = y^+ = 0$

$$y_1 + iy_2 = \frac{\sin \theta e^{i\varphi}}{(1 + \cos \theta)} = \tan \frac{\theta}{2} e^{i\varphi} \quad (2.10)$$

$$dy_1^2 + dy_2^2 = \frac{d\theta^2 + \sin^2 \theta d\varphi^2}{(1 + \cos \theta)^2} \equiv \Omega^2 ds_{S^2}^2 \mathcal{E}(y_1, y_2) = \Omega^{-3} \mathcal{E}(\vec{n}) = (1 + \cos \theta)^3 \mathcal{E}(\vec{n})$$

Physically, we expect that our idealized calorimeters will measure positive energies. Therefore, the expectation values of $\mathcal{E}(\vec{n})$ should be non-negative. In quantum field theory the expectation value of the stress tensor can be negative in some spacetime region. However, in our case we are integrating the stress tensor along a light like direction. In a free field theory one can show that the expectation value

$$\int dy^- \langle T_{--} \rangle \geq 0 \quad (2.11)$$

is positive on any state [17].⁶ We expect that the same should be true in an interacting field theory. In appendix A we recall the argument in free field theories and give a handwaving argument suggesting that this should be true in general. We will later see that this condition implies interesting constraints on certain field theory quantities, so it would be nice to be able to give a more solid argument for the positivity of (2.11) than the one we give in the appendix.

Notice that the energy flux operators $\mathcal{E}(\theta)$ commute with each other since operators at different values of θ are separated by spacelike distances. This is most clear when we express the operators in terms of the y coordinates as in (2.6). Thus, we can certainly consider the probability that we measure specific energy functions $\mathcal{E}(\theta) = f(\theta)$ and derive the probability functional that governs the process. One can also impose some cuts on the energy distribution and compute such probabilities. This is done when jet cross sections are computed, as in [5]. In fact, a specific Feynman diagram with n particles coming out at angles $\theta_1, \dots, \theta_n$ gives a contribution to the case where the energy function $f(\theta)$ is a delta function localized at these points. The energy correlation functions we have defined correspond to average energies where we also allow extra particles that come out and do not go into the calorimeters we are choosing to focus on.

⁶The curved space analog of this condition has also been explored for free fields in curved space, since it plays a role in proving singularity theorems in general relativity.

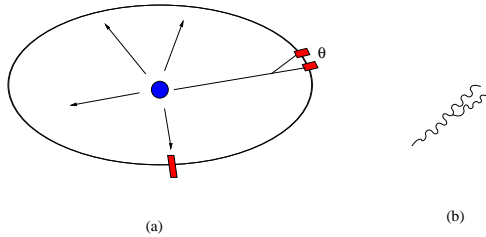


Figure 3: (a) Singularities in the energy correlation functions arise when we place two calorimeters very close to each other, at a small angle θ . (b) At the level of Feynman diagrams such singularities come from collinear radiation.

Besides putting a detector at infinity that measures energy we can also put a detector that measures charge. In that case we have the charge flux operator

$$\mathcal{Q}(\vec{n}) = \lim_{r \rightarrow \infty} r^2 \int_{-\infty}^{\infty} dt n^i j_i(t, r\vec{n}) \quad (2.12)$$

where j is the current associated to a global $U(1)$ symmetry of the field theory. In the coordinates (2.2) this becomes $\mathcal{Q}(y^1, y^2) = \int dy^- j_-(y^-, y^+ = 0, y^1, y^2)$. Under $SL(2, C)$ transformations \mathcal{Q} transforms as a field of conformal dimension two. We can similarly compute energy and charge correlation functions. One can also easily consider non-abelian global symmetries, and measure the components of various charges, as long as we do not put two charge insertions at the same point.

Now let us make some remarks on the operator ordering. Since the energy flux operators commute with each other for different θ , then, it does not matter how they are ordered. However, it is important that they are inserted between the operator, \mathcal{O} , that creates the state and the one annihilating it, as in (1.2). This is the standard ordering when we compute expectation values. If we use perturbation theory to compute them it is important that we do *not* use Feynman propagators since those are for time ordered situations. However, to do perturbation theory it is very convenient to use Feynman propagators. In such a case we have to be careful to remember that we should use the in-in [18, 19] formalism to evaluate the expectation value. This consists in choosing a contour that starts with the initial state, goes forward in time to the times where the stress tensor operators are evaluated and then goes backwards in time.

In a conformal field theory we could also consider the following. Minkowski space can be mapped to a finite region of $R \times S^3$. In fact, $R \times S^3$ can be split into an infinite number of regions, each of which is mapped to Minkowski space. In that case we can consider one of the regions as the original Minkowski space and the region immediately to the future as the region parametrizing the part of the Schwinger-Keldysh contour that goes back in time, as long as we transform the wavefunction of the in state in the bra appropriately. We found this picture useful for gaining intuition, but not particularly useful for doing computations.

2.2 Small angle singularities and the operator product expansion

The energy correlation functions develop singularities when two of the energy operators

are evaluated at very similar angles $\theta_1 \sim \theta_2$, see 3 a. Such singularities are related to collinear radiation. To leading order in the gauge theory coupling $\lambda = 4\pi\alpha_s N$ the leading singularity goes like $\mathcal{E}(\theta_1)\mathcal{E}(\theta_2) \sim \frac{C\lambda}{\theta_{12}^2}$ [7] and it comes from a Feynman diagram like the one shown in 3 b.

It is clear that such a limit should be characterized by some sort of operator product expansion. In this section we will make some remarks on the type of operators that appear in this expansion.

It is simpler to think about the problem in the y^μ coordinates introduced in (2.2). We should, then, compute the OPE of operators of the form

$$\mathcal{E}(y^1, y^2)\mathcal{E}(0, 0) \sim \int dy^- T_{--}(y^-, y^+ = 0, \vec{y}) \int dy'^- T_{--}(y'^-, y^+ = 0, \vec{0}) \quad (2.13)$$

The two operators are sitting at two different points in the transverse directions. We have set one at zero for convenience and the other at $\vec{y} = (y_1, y_2)$. Note that the distance between the two stress tensor insertions is $|\vec{y}|$ irrespective of the values of y^-, y'^- . This distance is spacelike, so one expects to be able to perform an operator product expansion when $\vec{y} \rightarrow \vec{0}$. Nevertheless, since the two stress tensors are sitting at two very different points in the y^- directions, the operators appearing in the OPE are not local operators. To leading order the operator is specified by two points that are light-like separated [20]. Such operators are useful for thinking about many high energy processes in QCD [21–23]. They are sometimes called “string operators” or “light ray” operators. Various “parton distribution” functions are defined in terms of matrix elements of such operators, see [22] for example. It is important to note that these operators are non-local along one light-like direction but they are perfectly local in all remaining three directions.

In order to characterize these non-local operators it is useful to label them according to their transformation properties under the conformal group [24] (see [25] for a review). Let us define the twist generator to be $T = \Delta - j$, where j is the spin (really a boost generator) in the y^+, y^- plane.⁷ More explicitly, the twist transformation is $(y^+, y^-, \vec{y}) \rightarrow (\lambda^2 y^+, y^-, \lambda \vec{y})$. The spin is the transformation $(y^+, y^-, \vec{y}) \rightarrow (\eta y^+, \eta^{-1} y^-, \vec{y})$. As it is well known, at zero coupling, one can consider twist two operators which correspond to primary operators of higher spin. For example, if we have a scalar field, ϕ , in the adjoint representation, then we can schematically define the operators

$$\mathcal{U}_j = Tr \left[\phi \overset{\leftarrow}{\partial} \overset{\rightarrow}{\partial}^j \phi \right] \quad (2.14)$$

This is schematic because there is a precise combination of derivatives that makes it a conformal primary.⁸ Such conformal primaries exist only if j is even. One is sometimes interested in extending the definition of such operators to generic, real or complex, values of j . This problem was considered in detail in [24]. There, it was found that one could

⁷Note that we define j to be the spin in the y^+, y^- plane only, not the total spin. The spin in the transverse directions is another generator which does not appear in the definition of the twist.

⁸The precise form is $\mathcal{U}_j = \sum_{k=0}^j \frac{(-1)^k}{[k!(j-k)!]^2} Tr \left[(\partial_-^k \phi) \partial_-^{j-k} \phi \right]$, where ϕ is a scalar field [26–28].

start with the operators

$$\mathcal{U}(y^-, y'^-) = \text{Tr}[\phi(y^-)W(y^-, y'^-)\phi(y'^-)] = \text{Tr} \left[\phi(y^-) P e^{\int_{y^-}^{y'^-} A} \phi(y'^-) \right] \quad (2.15)$$

where W is an adjoint Wilson line along a null direction. All operators are inserted at the same values of y^+ , y^1 and y^2 (but of course, at different values of y^-). We can also replace ϕ by a fermion or a gluon operator F_{-i} . Under twist transformations y^- remains invariant but the transverse coordinates are rescaled. In the quantum theory this scaling transformation mixes the operator (2.15) with operators with other values of y^-, y'^- . By thinking about the action of the collinear conformal group (the $\text{SL}(2, R)$ set of transformations of x^-) it is possible to diagonalize the action of the twist generator. To leading order the operators are diagonalized by considering suitable combinations of these light-ray operators [24, 25]. These operators are labeled by their center of mass momentum k_- along the y^- direction and their spin. For our purposes we will be interested only in operators which are integrated over the the center of mass position along the y^- direction so that they carry zero momentum along y^- . In that case the operators of arbitrary spin constructed from scalar fields can be written as

$$\mathcal{U}_{j-1} = \int_{-\infty}^{\infty} dy^- \int_0^{\infty} \frac{du}{u^{j+1}} \text{Tr}[\phi(y^- + u)W(y^- + u, y^- - u)\phi(y^- - u)] \quad (2.16)$$

The subindex of \mathcal{U} denotes the total spin and j denotes the spin before we do the y^- integration. This is an expression that makes sense for arbitrary complex values of j . When j approaches an even integer we find a pole in j coming from a logarithmic divergence in the integral at small u of the form $\int \frac{du}{u}$. The coefficient of this divergent term contains the ordinary local operator (2.14), see [24, 25] for more details. There are similar expressions for operators constructed from two fermions or two Yang-Mills field strengths. One can compute the value of the twist for these operators and one finds [24] $\tau(j) = 2 + \gamma(j)$, where $\gamma(j)$ is the anomalous dimension. One can also consider higher twist operators which contain more field insertions or extra derivatives with respect to the transverse direction or y^+ . In that case, in order to diagonalize the matrix of anomalous dimensions, it is not enough to give the total spin of the operator. Nevertheless, this can be done, see [25].

The OPE has the schematic form

$$\mathcal{E}(\vec{y})\mathcal{E}(\vec{0}) \sim \int dy^- T_{--}(y^-, \vec{y}) \int dy'^- T_{--}(y'^-, 0) \sim \sum_n |\vec{y}|^{\tau_n - 4} \mathcal{U}_{j-1, n} |_{j=3} \quad (2.17)$$

where the sum is over all operators which are local in y^+, \vec{y} , but not necessarily local in y^- , which have total spin $j-1 = 2$, (or $j = 3$) and twist τ_n . The spin is determined since the total spin of the left hand side is one for each of the two energy insertions. Equation (2.17) is schematic because we have not explicitly indicated the fact that the operators in the right hand side could carry spin in the transverse directions. A more precise expression has the form

$$\mathcal{E}(\vec{y})\mathcal{E}(\vec{0}) \sim \sum_{k, n} y^{(i_1 \dots i_k)} |y|^{\tau_{n, k} - k - 4} \mathcal{U}_{(i_1 \dots i_k); j-1; n} |_{j=3} \quad (2.18)$$

where we have now considered operators that carry spin in the transverse directions, the indices i_1, \dots, i_k are symmetric and traceless.

Among the operators which have twist two at zeroth order there are only a few that have $j = 3$. For example, in QCD there are only two, a bilinear in fermions and a bilinear in the gluon field strength. Thus, for the given spin we are considering ($j = 3$) we will have to diagonalize a finite matrix of anomalous dimensions.

In summary: *The small angle behavior of the energy correlation functions is determined by the spin $j = 3$ non-local operators that appear in the OPE*

$$\langle \mathcal{E}(\theta_1) \mathcal{E}(\theta_2) \dots \rangle \sim \sum_n |\theta_{12}|^{\tau_n - 4} \langle \mathcal{U}_{3-1,n}(\theta_2) \dots \rangle \quad (2.19)$$

where the dots denote other energy insertions and $|\theta|$ is the angle between the two energy insertions that are getting close to each other. The sum over n runs over all the higher twist operators that can appear. We will see that in $\mathcal{N} = 4$ super Yang Mills these operators develop large anomalous dimensions at strong coupling.

Note that the spin symmetry in the y^+, y^- plane, that we used to select the operators that contribute, is the dilatation symmetry of the original Minkowski space. This symmetry ensures that the energy correlation functions scale properly as we rescale the total energy (or rescale the variables x^μ). In other words, there can be no anomalous dimensions under total energy rescalings since that would conflict with energy conservation. This is the physical reason why we are forced to select particular operators in this OPE.

In the case of QCD the small angle behavior of energy correlation functions was computed a long time ago in [29, 30] using a slightly different language. They also needed to include the effects of the running coupling.

Let us now turn to the case of $\mathcal{N} = 4$ super Yang Mills at weak coupling. The weak coupling computation of the leading twist anomalous dimensions was done in [31, 32] (see also [33, 34]). We should consider operators which are invariant under all the symmetries that leave the particular component of the stress tensor in (2.18) invariant. These include the $SO(6)$ R-symmetry and a parity symmetry. We can classify the operators according to their transformation properties under the $SO(2)$ group that transforms the transverse coordinates. All operators are made out of a pair of scalars, fermions or gauge field strengths. The local operators with zero transverse spin in $SO(2)$ and spin j (j even) in the $+ -$ directions are [31]

$$Tr \left[\phi \overleftrightarrow{\partial}_-^j \phi \right], \quad Tr \left[F_{-i} \overleftrightarrow{\partial}_-^{j-2} F_{-i} \right], \quad Tr \left[\psi \Gamma_- \overleftrightarrow{\partial}_-^{j-1} \psi \right] \quad (2.20)$$

Supersymmetry relates these three towers of operators. Since supersymmetry carries spin, the various members of the supermultiplet have different spin. However, the anomalous dimension for all the members of the supermultiplet is the same and it is given by a function which has the weak coupling expansion [31, 35]

$$\gamma(j) = \frac{\lambda}{2\pi^2} [\psi(j-1) - \psi(1)] + \dots \quad (2.21)$$

where $\psi = \Gamma'(z)/\Gamma(z)$. This was computed also to two and three loops in [32]. The fact that $\gamma(j=2) = 0$ corresponds to the fact that the stress tensor is not renormalized.

Since we are interested in operators with a definite spin, we conclude that the three operators that diagonalize the anomalous dimension matrix are in three different multiplets. For spin three operators we have the anomalous dimensions [31]

$$\begin{aligned} \tau_1 - 2 &= \gamma(j=3), & \tau_2 - 2 &= \gamma(j=5), & (2.22) \\ \tau_3 - 2 &= \gamma(j=7) \tau_1 - 2 = \frac{\lambda}{2\pi^2}, & \tau_2 - 2 &= \frac{11\lambda}{12\pi^2}, & \tau_3 - 2 &= \frac{137\lambda}{120\pi^2} \end{aligned}$$

where we just gave the first order expression. We see from (2.22) and (2.21) that all three anomalous dimensions in (2.22) are positive and $\tau_1 - 2$ is the smallest one which will give us the leading order singularity. However, for weak coupling all three contributions are similar.

In addition to the operators we discussed, we can also have operators which have non-zero transverse spin. At twist two, the only one consistent with the symmetries is the spin two operator

$$\mathcal{U}_{(il);j} = Tr \left[F_{-(i} \overleftrightarrow{\partial}^{j-2} F_{-l)} \right] \quad (2.23)$$

where the indices $i, l = 1, 2$ are symmetrized and traceless. In the the $\mathcal{N} = 4$ theory these operators are in the same supermultiplet as the ones considered above [36]. For this reason their anomalous dimension is also given in terms of the same formula

$$\tilde{\tau}_j - 2 = \gamma(j+2), \quad \tilde{\tau}_3 = 2 + \frac{11\lambda}{12\pi^2} \quad (2.24)$$

Thus we expect to have a small angle singularity of the form

$$\langle \mathcal{E}(\vec{y}) \mathcal{E}(0) \dots \rangle \sim \sum_{a=1}^3 |y|^{-2+(\tau_a-2)} c_a \langle \mathcal{U}_a \dots \rangle + y^{(i} y^{l)} |y|^{-4+(\tilde{\tau}_3-2)} \tilde{c} \langle \mathcal{U}_{(il)} \dots \rangle \quad (2.25)$$

where the operators \mathcal{U}_a are the linear combinations that diagonalize the anomalous dimension matrix for the operators with zero transverse spin. c_a and \tilde{c} are coefficients that can be obtained by performing the operator product expansion explicitly. These constants are independent of the state for which we compute the energy correlation. Of course, the terms $\langle \mathcal{U}_a \dots \rangle$ and $\langle \mathcal{U}_{(il)} \dots \rangle$ do depend on the state on which we compute the energy correlation function. The coefficients c_a, \tilde{c} start at order λ at weak coupling since it is easy to check explicitly that at tree level there is no contribution to the operator product expansion of two energy flux operators.

In QCD one can do a similar analysis, including the effects of the beta function, see [30]. In that case, the operator made out from scalars in (2.20) does not contribute.

Having done one OPE, we could also do a further OPE of the resulting operator with a third energy flux operator. That would give an operator of total spin $j-1=3$, or $j=4$, and so on. More generally, we can consider the case where n energy operators come close together. If we keep the ratios of angles between these n points fixed, then the small angle

behavior is given by the anomalous dimension of the operator of spin $j = n + 1$. The structure of a jet at weak coupling is largely controlled by these operator product expansions.

Note that, at weak coupling, after we consider the effects of the anomalous dimensions, the energy correlators have small angle singularities that are integrable. In fact, if we do the integral over a small angle θ_0 of the energy two point function we find schematically

$$\int d^2\theta \mathcal{E}(\theta) \mathcal{E}(0) \sim \int_0^{\theta_0} d^2\theta \frac{\lambda}{\theta^{2-\gamma_*\lambda}} \sim (\theta_0)^{\gamma_*\lambda} \quad (2.26)$$

where the anomalous twist of the spin three operator is $\tau - 2 = \gamma_*\lambda$ and γ_* is a numerical constant. This expression is schematic because at weak coupling we have to include all the terms in (2.25). If θ_0 is fixed and $\lambda \rightarrow 0$ then we see that the integral gives a finite answer. This is to be expected since the total integral of one energy insertion over the whole sphere should give the total energy, independently of λ .⁹ The fact that we get a finite contribution from this region is consistent with the idea that the energy is going out in localized jets. We can also estimate the angular size of jets, by finding a θ_0 in such a way that we get a fixed fraction, f , of the total energy in the jet. This gives an estimate $\theta_0 \sim e^{-c/\lambda}$, where c depends on f . This was originally discussed in [5], see also [30] for a more detailed discussion.

We could also do an OPE of two charge operators, each of which has spin zero, after we integrate the spin one current over y^- (2.12). In this case we get operators with total spin $j - 1 = 0$, or $j = 1$. Some of these have negative anomalous dimensions. In fact, we expect that charge correlators would be more singular at small angles due to the fact that a gluon can create a pair of oppositely charged particles fairly easily and there is no reason that we couldn't get a divergence when we integrate the charge correlator at small θ .

Finally, we should mention that in QCD the energy-energy correlation two point function was computed for all angles in,¹⁰ [7, 38–41]. It was also compared to experiment in [8], where it was used as a way to measure α_s .

2.3 Energy flux one point functions

In this section we will make some simple and general remarks about the energy flux one point function

$$\langle \mathcal{E}(\vec{n}) \rangle = \frac{\langle 0 | \mathcal{O}_q^\dagger \mathcal{E}(\vec{n}) \mathcal{O}_q | 0 \rangle}{\langle 0 | \mathcal{O}_q^\dagger \mathcal{O}_q | 0 \rangle} \quad (2.27)$$

These one point functions are determined up to a few coefficients by Lorentz symmetry, even in non-conformal theories. Here we will consider these in the CFT context in order to make contact with other results in conformal field theories.

The energy flux one point function (2.27) amounts to computing a three point function in the CFT. Three point functions in a generic CFT are determined up to a few numbers by conformal symmetry [42–44].

⁹Here we are also assuming that the energies are locally positive. For charge correlators we cannot make the same argument because the charge can be positive or negative.

¹⁰The results presented in the following references show some disagreements. For a detailed comparison between these results see [37]. We thank S. Catani for pointing this out to us.

Let us start with the case that we create the external state with a scalar operator with energy q and zero momentum. Strictly speaking such an operator is not a localized insertion. Thus, more precisely, we will be considering operators of the form

$$\mathcal{O}_q \equiv \int d^4x \mathcal{O}(x) e^{-iqx_0} \exp \left\{ -\frac{x_0^2 + x_1^2 + x_2^2 + x_3^2}{\sigma^2} \right\}, \quad q\sigma \gg 1 \quad (2.28)$$

where the last inequality ensures that the operator is localized, has finite norm and has four momentum approximately $\tilde{q}^\mu = (q, \vec{0}) + o(1/\sigma)$. In particular we have $q^0 \sim q$. Once we know this precise form of the operator we see that we can also write it in other coordinate systems by performing the suitable conformal transformation and taking into account the conformal transformation properties of $\mathcal{O}(x)$.

In what follows we will consider field theory states produced by scalar operators, $\mathcal{O} \sim \mathcal{S}$, conserved currents $\mathcal{O} \sim \epsilon_i j_i$, and the stress tensor, $\mathcal{O} \sim \epsilon_{ij} T_{ij}$. In all cases we consider states with essentially zero spatial momentum as in (2.28). The case where q^μ is a generic four vector can be obtained by performing a simple boost of the configurations we discuss.

In the case that we insert a scalar operator it is clear by $O(3)$ symmetry that the energy one point function is constant on the two sphere. In addition the integral over the angles should give the total energy. Thus, for a scalar operator we have

$$\langle \mathcal{E}(\vec{n}) \rangle = \frac{q}{4\pi} \quad (2.29)$$

Even though we know the answer already, it is possible to do the calculation explicitly by writing down the unique general expression for the three point function of two scalars and the stress tensor [43]. Its normalization is fixed by a Ward identity in terms of the two point function of the two scalars. This Ward identity is another version of the energy conservation argument that we used above. Writing down the three point function and doing the integrals in the limit (1.1) we indeed obtain (2.29). One has to be careful about the operator ordering. In appendix C we do this explicitly.

We now turn to the case where the external perturbation couples to a conserved current in the CFT. In that case the operator is given by $\mathcal{O}_{\epsilon, q} \sim \epsilon^\mu j_\mu(q)$ where ϵ^μ is a constant polarization vector. Due to the current conservation condition we can identify $\epsilon^\mu \sim \epsilon^\mu + \lambda q^\mu$. So we can choose ϵ to point in the spacelike directions. In this case $O(3)$ symmetry and the energy conservation condition constrain the form of the one point function to

$$\langle \mathcal{E}(\vec{n}) \rangle = \frac{\langle 0 | (\epsilon^* \cdot j^\dagger) \mathcal{E}(\vec{n}) (j \cdot \epsilon) | 0 \rangle}{\langle 0 | (\epsilon^* \cdot j^\dagger) (j \cdot \epsilon) | 0 \rangle} = \frac{q}{4\pi} \left[1 + a_2 \left(\frac{|\vec{\epsilon} \cdot \vec{n}|^2}{|\vec{\epsilon}|^2} - \frac{1}{3} \right) \right] = \frac{q}{4\pi} \left[1 + a_2 \left(\cos^2 \theta - \frac{1}{3} \right) \right] \quad (2.30)$$

where θ is the angle between between the point on the S^2 , labeled by n^i , and the direction of the polarization vector ϵ^i .

The fact that we have one free parameter is in agreement with the general analysis of the three point function of two conserved currents and the stress tensor. In fact in [43] it was shown that the three point function is determined by conformal symmetry up to two parameters and one of them is fixed by the Ward identity of the stress tensor.

Note that a_2 in (2.30) obeys a constraint that comes from demanding that the expectation value of the energy $\mathcal{E}(\theta)$ is positive, see (2.11). This condition leads to the constraint

$$3 \geq a_2 \geq -\frac{3}{2} \quad (2.31)$$

This one point function was computed for the electromagnetic current in QCD in [7]. To first order in α_s the result is

$$a_2 = -\frac{3}{2} + \frac{9\alpha_s}{2\pi} + \dots \quad (2.32)$$

To the order written in (2.32) we can approximate the QCD computation by a conformal field theory with the value of the coupling set by the energy of the process $\alpha_s = \alpha_s(|q^0|)$.

In the application to e^+e^- collisions that produce a gauge boson which in turns couples to a current the polarization vector of the current depends on the polarization states of the e^+ and e^- as well as the type of gauge boson we are considering (γ or Z , Z' , etc). In the case that we consider unpolarized electrons we can express the answer in terms of the angle with respect to the beam axis, θ_b . ($\cos \theta_b = \vec{n} \cdot \hat{z}$ where \hat{z} is the beam axis). The polarization vectors for the current are orthogonal to the beam direction and we should average over them. After doing this average, we find that (2.30) becomes

$$\langle \mathcal{E}(\vec{n}) \rangle = \frac{\sum_s \langle 0 | (\epsilon_s^* \cdot j^\dagger) \mathcal{E}(\vec{n}) (j \cdot \epsilon_s) | 0 \rangle}{\sum_s \langle 0 | (\epsilon_s^* \cdot j^\dagger) (j \cdot \epsilon_s) | 0 \rangle} = \frac{q}{4\pi} \left[1 + a_2 \left(\frac{1}{2} \sin^2 \theta_b - \frac{1}{3} \right) \right] \quad (2.33)$$

where we sum over polarization vectors transverse to the beam. For a current that couples to free fermions we find the familiar $(1 + \cos^2 \theta_b)$ distribution, as we can check from the leading order QCD result (2.32).

For a current that couples to free complex bosons of charges q_i^b and Weyl fermions of charges q_i^{wf} we get

$$a_2^{\text{free}} = 3 \frac{\sum_i (q_i^b)^2 - (q_i^{wf})^2}{\sum_i (q_i^b)^2 + 2 (q_i^{wf})^2} \quad (2.34)$$

where we sum over both left and right Weyl fermions. Note that the case where we only have bosons saturates the upper bound in (2.31) and free fermions saturate the lower bound in (2.31). In fact, going back to (2.33) we see that we get the well known distributions proportional to $\sin^2 \theta_b$ or $(1 + \cos^2 \theta_b)$ for free bosons and fermions respectively.

We can consider a similar problem now in an $\mathcal{N} = 1$ superconformal theory. If the current is a global symmetry that commutes with supersymmetry (a non-R symmetry) then one can see that $a_2 = 0$. In a free supersymmetric theory we see from (2.34) that the bosons and Weyl fermions cancel each other. For an interacting theory this follows from the fact that such a current is in the same multiplet as a scalar operator, and for a scalar operator we do not have any arbitrary parameters [45]. Thus the value of a_2 is fixed by superconformal symmetry. However, since we got $a_2 = 0$ in a free theory, we have $a_2 = 0$ for any global symmetry of a SCFT.

On the other hand we can get a non-zero value of a_2 if we consider the R current.¹¹ The R current is in a different supermultiplet. In fact, it is in a supermultiplet with the stress tensor. All three point functions among elements of this supermultiplet are determined by two numbers, c and a [45]. These numbers also characterize the anomalies of the R current, which are encoded in parts of the jjj and jTT three point functions [46, 45]. They also contribute to the conformal anomaly on a general background,

$$T_{\mu}^{\mu} = \frac{c}{16\pi^2} W_{\mu\nu\delta\sigma} W^{\mu\nu\delta\sigma} - \frac{a}{16\pi^2} E, \quad E = R_{\mu\nu\delta\rho} R^{\mu\nu\delta\rho} - 4R_{\mu\nu} R^{\mu\nu} + R^2 \quad (2.35)$$

where W is the Weyl tensor and E the Euler density. The c coefficient is the only constant that appears in the two point functions of the currents and the stress tensor [45]. Thus c appears in the part of the three point function that is fixed by the Ward identity. The coefficient a_2 is given by a linear combination of a and c . The particular linear combination is independent of the theory. It is fixed by supersymmetry. We can compute the precise combination by considering the particular case of free field theories. As we explain in more detail below we find that

$$\langle \mathcal{E}(\theta) \rangle = 1 + 3 \frac{c-a}{c} \left(\cos^2 \theta - \frac{1}{3} \right) \quad (2.36)$$

This formula was obtained as follows. We used that for a free supersymmetric theory with n_V vector multiplets and n_S chiral multiplets we have $48a = 9n_V + n_S$, and $24c = 3n_V + n_S$ [46–48]. The vector multiplet has one Weyl fermion of charge 1 and the scalar multiplet in a free theory has a Weyl fermion of charge $-1/3$ and a boson of charge $2/3$. Then using (2.34) we obtain (2.36). Note that even though we used free field theories to fix the numerical coefficients, the final result (2.36) is true for a general interacting $\mathcal{N} = 1$ SCFT.

In $\mathcal{N} = 4$ super Yang Mills $a = c$ and the result for the one point function is spherically symmetric. Of course this is not a surprise since $U(1)$ subgroups of the $SO(6)$ symmetry group can also be viewed as global symmetries from the point of view of $\mathcal{N} = 4$ written as an $\mathcal{N} = 1$ theory. Thus, in $\mathcal{N} = 4$ super Yang mills the result is independent of the coupling.

The positivity constraint (2.31), together with (2.36) gives $\frac{3c}{2} \geq a \geq 0$.

We can also consider the energy one point function in the case that the state is created by the stress tensor. As above, we take the momentum of the inserted operator in the time like direction. Then the operator that we are considering is characterized by a symmetric polarization tensor ϵ^{ij} which we take to have indices in the purely spacelike directions by using the conservation equations. Since the stress energy tensor is traceless, we also take $\epsilon^{ii} = 0$. By $O(3)$ invariance we see that the most general form of the three point function is

$$\langle \mathcal{E}(\theta) \rangle = \frac{\langle 0 | \epsilon_{ij}^* T_{ij} \mathcal{E}(\theta) \epsilon_{lk} T_{lk} | 0 \rangle}{\langle 0 | \epsilon_{ij}^* T_{ij} \epsilon_{lk} T_{lk} | 0 \rangle} = \frac{q^0}{4\pi} \left[1 + t_2 \left(\frac{\epsilon_{ij}^* \epsilon_{il} n_i n_j}{\epsilon_{ij}^* \epsilon_{ij}} - \frac{1}{3} \right) + t_4 \left(\frac{|\epsilon_{ij} n_i n_j|^2}{\epsilon_{ij}^* \epsilon_{ij}} - \frac{2}{15} \right) \right] \quad (2.37)$$

We see that we have two undetermined coefficients. This agrees with the general analysis of stress tensor three point functions in [43], where they found that conformal

¹¹We thank Scott Thomas for pointing this out.

symmetry determines the three point function of the stress tensor up to three coefficients, one of which is fixed by a Ward identity. We have chosen the constants in the last two terms in (2.37) in such a way that the corresponding terms integrate to zero on the sphere.

By demanding that $\langle \mathcal{E} \rangle$ is positive we get the constraints

$$\begin{aligned} \left(1 - \frac{t_2}{3} - \frac{2t_4}{15}\right) &\geq 0 \\ 2\left(1 - \frac{t_2}{3} - \frac{2t_4}{15}\right) + t_2 &\geq 0 \quad \frac{3}{2}\left(1 - \frac{t_2}{3} - \frac{2t_4}{15}\right) + t_2 + t_4 &\geq 0 \end{aligned} \quad (2.38)$$

We obtain these constraints by using $O(3)$ invariance in (2.37) to set $\vec{n} = \hat{z}$. We then view the resulting equation as a bilinear form on the space of ϵ 's. This space can be divided into three orthogonal parts according to their $O(2)$ invariance properties (the spin of ϵ along the \hat{z} axis). We have an $SO(2)$ scalar, a vector and a symmetric traceless tensor. On each of these subspaces we get each of the constraints (2.38). Each of these limits is saturated in a free theory with no vectors, no fermions or no bosons respectively. The fact that the first equation is saturated in a theory without vectors is clear. In that case, if we consider a stress tensor insertion with spin +2 in the \hat{z} direction we cannot have emission of bosons or fermions in the \hat{z} direction due to the orbital angular momentum wavefunctions. It is also possible to write a general bound on the two coefficients that appear in the conformal anomaly (2.35), see appendix C.

In an $\mathcal{N} = 1$ supersymmetric theory we find that

$$t_2 = 6(c - a)/c, \quad t_4 = 0 \quad (2.39)$$

By requiring that (2.37) is positive for all choices of traceless ϵ_{ij} we find

$$\frac{3}{2}c \geq a \geq \frac{c}{2} \quad (2.40)$$

Of course $c > 0$ due to the positivity of the two point functions. The bounds are saturated by free theories with only vector supermultiplets (upper bound) or only chiral supermultiplets (lower bound). It is interesting that the lower bound that we obtain in this way is precisely the same as the bound obtained in [49] (see also [50]) based on causality for a gravity theory that contains only the Einstein term and a R^2 term.¹² In the theory considered in [49] one would also have $t_4 = 0$, though it is not clear whether it corresponds to any dual quantum field theory. Here we have only used general field theory considerations.

For a non-supersymmetric theory it is also possible to derive a bound from (2.38). As explained in appendix C we find

$$\frac{31}{18} \geq \frac{a}{c} \geq \frac{1}{3} \quad (2.41)$$

where the lower bound is saturated by a free theory with only scalar bosons and the upper bound by a free theory with only vectors. Note that the bound in supersymmetric theories (2.40) is more stringent than in non-supersymmetric theories (2.41). Let us also

¹²In order to see this one has to set $\lambda_{GB} \rightarrow 9/100$ (the bound in [50]) into the expressions for a and c [51] (see eq. (5.1) of [50]).

add that the results from appendix C also allow us to calculate this bound for $\mathcal{N} = 2$ supersymmetric theories. In this case we can obtain the bound by taking the operator \mathcal{O} to be one of the $SU(2)$ R-symmetry generators and demanding the the energy one point function is positive. The result is in this case

$$\frac{5}{4} \geq \frac{a}{c} \geq \frac{1}{2} \quad (2.42)$$

This is a smaller window than for the $\mathcal{N} = 1$ case, as expected. The upper bound corresponds to a free theory with vector supermultiplets only while the lower bound corresponds to a free theory with hypermultiplets only. This agrees with results in [52].

We can make similar remarks for operators that involve charge correlations. For example, we could consider a theory with an $SU(2)$ global symmetry and then select one $U(1) \subset SU(2)$ to form the charge flow operator \mathcal{Q} that we measure at infinity. We could consider a charged state state created by the current $\epsilon \cdot J^+$, where the plus indicates that it carries charge plus one. As in the energy correlations the charge correlations have a form

$$\langle \mathcal{Q}(\vec{n}) \rangle = \frac{\langle 0 | \vec{\epsilon}^* \cdot \vec{J}^- \mathcal{Q}(\vec{n}) \vec{\epsilon} \cdot \vec{J}^+ | 0 \rangle}{\langle 0 | \vec{\epsilon}^* \cdot \vec{J}^- \vec{\epsilon} \cdot \vec{J}^+ | 0 \rangle} = \frac{1}{4\pi} \left[1 + \tilde{a}_2 \left(\frac{|\vec{\epsilon} \cdot \vec{n}|^2}{|\vec{\epsilon}|^2} - \frac{1}{3} \right) \right] \quad (2.43)$$

Again, the coefficient \tilde{a}_2 is related to the fact that there are two possible (parity preserving) structures for the three point function of three currents [42, 43]. One of them is fixed by the Ward identities in terms of the two point functions, a fact we used in (2.43). We note that in a supersymmetric theory where these currents are global symmetries $\tilde{a}_2 = 0$. One can show this as follows. First note from [45] that there is only one parity preserving structure for current three point functions in supersymmetric theories. This means that the value of \tilde{a}_2 is fixed by supersymmetry. One can show that it vanishes by computing it in a particular theory, such as a free theory or $\mathcal{N} = 4$ super Yang Mills at strong coupling.

There are some cases where there are parity odd structures that can contribute. Such parity odd parts of three point functions are related to anomalies. For example, in the case of three currents these are related to the usual anomaly [42]. Consider the case that we have an external state produced by a current and we measure a charge one point function distribution. For example, we can consider a $U(1)$ current that has a cubic anomaly. A concrete example is the R current in superconformal theories. We consider a state obtained by acting with this current on the vacuum and we measure the charge flux far away. We find

$$\langle \mathcal{Q}(\vec{n}) \rangle = \frac{\langle 0 | \vec{\epsilon}^* \cdot \vec{j}^\dagger \mathcal{Q}(\vec{n}) \vec{\epsilon} \cdot \vec{j} | 0 \rangle}{\langle 0 | \vec{\epsilon}^* \cdot \vec{j}^\dagger \vec{\epsilon} \cdot \vec{j} | 0 \rangle} = i \alpha \epsilon_{jlk} \epsilon_j^* \epsilon_l n_k \sim \alpha \cos \chi \quad (2.44)$$

Note that this is non-zero only if ϵ is complex. This happens, for example, when we consider a circularly polarized state. Then χ is the angle between the direction of the spin of the current and the calorimeter. This leads to a charge flow asymmetry. Such asymmetries are extensively studied in e^+e^- collisions that produce a Z boson that then decays. Here we are pointing out that the charge flow asymmetries are related to the anomaly. Of course the full electroweak symmetry is not anomalous. But if one focuses only in decays of the Z into leptons, then the fact that the purely leptonic theory is anomalous leads to the

charge flow asymmetry. The fact that tree level processes plus unitarity fix the anomaly was pointed out in [53].

The three point function of two stress tensors and a current has a term that reflects the mixed gravitational anomaly [54]. Consider the case where the stress tensor creates the state and we measure the charge distribution of the current that has a mixed anomaly. The charge distribution has the structure

$$\langle \mathcal{Q}(\theta) \rangle = \frac{\langle 0 | \epsilon_{ij}^* T_{ij} \mathcal{Q}(\theta) \epsilon_{lk} T_{lk} | 0 \rangle}{\langle 0 | \epsilon_{ij}^* T_{ij} \epsilon_{lk} T_{lk} | 0 \rangle} = i\beta \frac{\epsilon_{ljk} \epsilon_{rl}^* \epsilon_{sj} n^r n^s n^k}{|\epsilon_{ij}|^2} \quad (2.45)$$

where β is related to the anomaly coefficient. For a supersymmetric theory, and for \mathcal{Q} given by the R-current, we have that $\beta \sim (a - c)$ [46, 54]. Notice that there is, in principle, another tensor structure consistent with $O(3)$ symmetry that could have contributed to (2.45), namely $\epsilon_{ljk} \epsilon_{rl}^* \epsilon_{rj} n^k$. This term is, however, absent from the three point function once conformal symmetry and the Ward identities are imposed [54]. Thus in a theory with a gravitational mixed anomaly there is charge asymmetry for the states produced by the graviton.

2.4 Relation to deep inelastic scattering

In this section we explore the relation between the energy correlation functions and the deep inelastic scattering cross sections.

The deep inelastic cross section for the scattering of an electron from a proton can be factorized into the electromagnetic process and the strong interactions process. At lowest order in the electromagnetic coupling, but exactly in α_s , the strong interactions part of the cross section can be written in terms of the expectation value of two currents in the the state of the target (which is traditionally a proton, but can be generalized to any other particle)

$$\begin{aligned} \tilde{W}^{\mu\nu} &= \int d^4 y e^{iqy} \langle p | J^\mu(0) J^\nu(y) | p \rangle = \\ &= \tilde{F}_1(x, q^2/p^2) \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + \frac{2x}{q^2} \tilde{F}_2(x, q^2/p^2) \left(p_\mu + \frac{q_\mu}{2x} \right) \left(p_\nu + \frac{q_\nu}{2x} \right) \end{aligned} \quad (2.46)$$

$$\text{where} \quad x \equiv -\frac{q^2}{2p \cdot q}, \quad q^2 > 0$$

We are imagining that we have a plane wave state in the y coordinates with timelike momentum $p^2 < 0$. The tensor (2.46) is nonvanishing only if we create a timelike state $s = -(q+p) \geq 0$ with the current. For these values of p and q our definition of $\tilde{W}^{\mu\nu}$, (2.46), coincides with the ordinary one [55], which involves a commutator of the currents.

We would like to relate these formulas to the ones appearing in the energy correlators. Let us consider the charge operator Q evaluated in the y coordinates

$$\mathcal{Q}(\vec{y}) = \int dy^- j_-(y^+ = 0, y^-, \vec{y}) \quad (2.47)$$

where \vec{y} denotes two transverse dimensions. These two transverse dimensions are related to the angles on the two sphere by (2.10). In the y coordinates, we can Fourier

transform this operator. We, then, have something similar to the current appearing above, except that the current in (2.46) is in momentum space also in the y^+ direction. Note also that $q_- = 0$, due to the y^- integral in the definition of the charge flux operator \mathcal{Q} . Since q_- is zero, we find that $q^2 = (\vec{q})^2 > 0$ and independent of q_+ . However x depends on q_+ since

$$\frac{1}{x} = \frac{-2p \cdot q}{q^2} = \frac{-(p+q)^2 + p^2 + q^2}{q^2} = \frac{4p_- q_+}{q^2} + \dots \quad (2.48)$$

where the dots indicate terms that are independent of q_+ . In order to produce the $\delta(y^+)$ that is present in the charge flux operator we need to integrate over q_+ . This integral translates into an integral over x . The range of integration can be determined by the condition that $-(p+q)^2 \geq 0$. Thus we end up integrating between $x = 0$ and x_{\max} with $1/x_{\max} = 1 + p^2/q^2$. In the limit $p^2/q^2 \rightarrow 0$ we get the usual boundary $x = 1$.

We then have the following relation between the two quantities

$$\begin{aligned} \int d^2 \vec{y} e^{i\vec{q} \cdot \vec{y}} \langle \mathcal{Q}(\vec{0}) \mathcal{Q}(\vec{y}) \rangle_p &= \int_{-\infty}^{\infty} \frac{dq_+}{2\pi} W_{--}(q_+, q_- = 0, \vec{q}; p) \\ &= \frac{(-p_-)}{4\pi} \int_0^{x_{\max}} \frac{dx}{x} F_2(x, q^2/p^2) \end{aligned} \quad (2.49)$$

This is a particular moment of the parton distribution functions. More precisely it is the moment $M_1^{(2)}$. As it is well known, the even moments M_{2k} can be expressed in terms of the expectation values of local operators with spins $j = 2k$ [55], via a dispersion relation argument. In fact, the moments $M_j^{(2)}$ can also be expressed in terms of the expectation values of the non-local light-ray operators with spin j for any j , see [56] for a general discussion.

In (2.49) the charge correlation is evaluated on a state with definite momentum in the y coordinates. This implies, in particular, that the charge two point function is also translation invariant in the transverse space, $\langle \mathcal{Q}(\vec{y}) \mathcal{Q}(y') \rangle_p = \langle \mathcal{Q}(\vec{y} - \vec{y}') \mathcal{Q}(0) \rangle_p$.

In this article we have been mainly considering states which are in momentum eigenstates in the x coordinates, related to the y coordinates via (2.2). This does not lead to momentum eigenstates in the y coordinates. However, they do have definite momentum in the p_- direction. To the extent that we can neglect other components of the momentum in the y coordinates we see that the charge correlator has a simple relation to the deep inelastic scattering amplitude and the ordinary parton distribution functions. In the general case we will need to evaluate expectation values of the form $\langle p' | J J | p \rangle$. These require generalized parton distribution functions [57]. Thus, if we have a state with definite momentum in the original x -coordinates, we will have a supersposition of momenta in the y coordinates and the charge two point function will be related to integrals over generalized parton distribution functions. We will not write a detailed expression here.

Notice that the integral over x is divergent at small x . We think that this is due to the fact that the integral over \vec{y} is also divergent for the charge correlator since the small angle singularity is not integrable. This divergence, though, is local in \vec{q} and can probably be extracted without changing the overall picture. We have not checked this in detail. This problem is not present if we consider the energy correlation functions and the relation to the deep inelastic amplitudes probed by gravitons. In that case all quantities are manifestly finite.

The fact that in our problem we do not have ordinary plane wave wavefunctions in y has an interesting consequence. It was shown in [2] that, in the gravity regime, the leading power of q , which governs the short distance behavior in \vec{y} , is controlled by a double trace operator. We will show below that this contribution is highly suppressed for operators that have definite momentum in x -space. We expect that this double trace contribution will also be suppressed at weak coupling when we consider plane wave states in x -space.

Of course, everything we said here can be repeated for the energy correlation function, except that we should consider a deep inelastic process where we scatter gravitons from the field theory excitations.

2.5 Energy correlations and the C parameter

Let us make here a side comment on the relation between the energy correlators and other usually considered event shape variables. Event shape variables are certain functions of the four momenta of the observed particles which are infrared safe. One concrete example is the C parameter, defined as [58]

$$C = \frac{3}{2E^2} \int d^2\Omega_1 d^2\Omega_2 \mathcal{E}(\vec{n}_1) \mathcal{E}(\vec{n}_2) \sin^2 \theta_{12} \quad (2.50)$$

where E is the total energy (and we assume that the total momentum vanishes). We see that the expectation value of C is given by an integral over the energy two point correlation function.

On the other hand, it is common to compute the cross section as a function of C (see for example [59]). This is just the probability of measuring various values of C , $\frac{d\sigma}{dC}$. This calculation involves more input than the two point correlation function, since we would need to know all the moments of C , $\langle C^n \rangle$, to reconstruct $\frac{d\sigma}{dC}$.

The point of this short remark is to stress that, even though the C parameter is given by a product of energies, the computation of the cross-section as a function of C involves knowledge of the n point energy-correlation functions. Of course, in practice, $\frac{d\sigma}{dC}$ is computed directly rather than going through the energy correlation functions.

3. Energy correlation functions in theories with gravity duals

In this section we consider energy correlation functions in conformal field theories that have gravity duals. We first start with some general remarks on the energy correlators and the basic ingredients necessary to calculate them. Then, we will present explicit calculations for the energy one point functions, which are given in terms of three point functions in the gravitational theories. Finally, we add a general prescription for computing arbitrary n point functions.

3.1 General remarks and basic ingredients of the calculation

The general prescription for computing correlation functions of local operators in the CFT using the gravity dual was derived in [16, 15]. Computations of the expectation values of the stress tensor for falling objects include [3, 60]. Since energy flux correlation functions

are given in terms of stress tensor correlators, we simply need to perform the integral over time and take the limit in (1.1). In order to simplify the computations it is useful to consider some coordinate changes.

Let us start by writing AdS_5 using the coordinates

$$-(W^{-1})^2 - (W^0)^2 + (W^1)^2 + (W^2)^2 + (W^3)^2 + (W^4)^2 = -1 \quad (3.1)$$

The boundary of AdS_5 corresponds to the region where $W^M \rightarrow \infty$. In that regime we can forget about the -1 in (3.1) and we recover the coordinates Z^M that we described around (2.3). It will be convenient to introduce three possible sets of coordinates which are natural from different points of view. The first two are

$$\begin{aligned} \text{Original :} \quad & \frac{1}{z} = W^{-1} + W^4, \quad W^\mu = \frac{x^\mu}{z}, \quad \mu = 0, 1, 2, 3 \\ \text{Easy :} \quad & \frac{1}{y_5} \equiv W^0 + W^3, \quad W^{-1} = -y^0, \quad W^4 = -y^3, \quad W_{1,2} = \frac{y_{1,2}}{y_5} \end{aligned} \quad (3.2)$$

Of course the metrics are simply

$$\begin{aligned} \text{Original :} \quad & ds^2 = \frac{dx^2 + dz^2}{z^2} \\ \text{Easy :} \quad & ds^2 = \frac{-dy^+ dy^- + dy_1^2 + dy_2^2 + dy_5^2}{y_5^2} \end{aligned} \quad (3.3)$$

It is also convenient to introduce a third set of coordinates, which is defined as follows. We first choose three coordinates describing the H_3 subspace $-(W^0)^2 + (W^1)^2 + (W^2)^2 + (W^3)^2 = -r^2$ for a fixed r^2 . The two other coordinates are chosen as

$$W^\pm = W^{-1} \pm W^4 \quad (3.4)$$

Then $r^2 = 1 - W^+ W^-$ and the metric is

$$\text{Hyperbolic :} \quad ds^2 = -dW^+ dW^- - \frac{1}{4} \frac{(W^- dW^+ + W^+ dW^-)^2}{1 - W^+ W^-} + (1 - W^+ W^-) ds_{H_3}^2 \quad (3.5)$$

The advantage of this coordinate system is that it makes the $SO(1,3)$ symmetry of the problem manifest. This $SO(1,3)$ symmetry are the isometries of H_3 . In addition, the dilatation symmetry in the original coordinates becomes a boost in the W^\pm plane, which is also a clear symmetry of the metric in this parametrization.

The surface that is at the boundary of four dimensional Minkowski space can be extended to the interior in a unique way so that it is invariant under the symmetries that preserve the boundary of Minkowski space. In fact, this surface is simply given by $W^+ = 0$.

The insertion of the stress tensor operator corresponds to a non-normalizable perturbation of the metric in the the bulk. It will be convenient to derive first the expressions for the momentum in the y coordinates introduced in (2.2). Since all the generators in (2.6) are given in terms of the integral of $T_{--}(y')$ over a line along y'^- , let us compute this first. We insert $T_{--}(y')$ on the boundary at $y'^+ = 0$ and $y'^1 = y'^2 = 0$ but at an arbitrary value

of y'^- . We denote the boundary points with primes and the bulk points without primes. This induces the following fluctuation in the metric of AdS_5 , $g_{MN} \rightarrow g_{MN} + h_{MN}$,

$$h_{MN} dx^M dx^M \sim (dy^+)^2 \frac{y_5^2}{[-y^+(y^- - y'^-) + y_1^2 + y_2^2 + y_5^2 + i\epsilon]^4} \quad (3.6)$$

We can now perform the integral over y'^- . We use the formula

$$\int_{-\infty}^{\infty} dy'^- \frac{1}{[y^+ y'^- + A + i\epsilon]^4} \sim \delta(y^+) \frac{1}{A^3} \quad (3.7)$$

Now, by a simple translation, we can set the energy operator at any other value of $y'^{1,2}$. We obtain

$$\begin{aligned} h_{MN} dX^M dX^N &\sim \delta(y^+) (dy^+)^2 \frac{y_5^2}{(y_5^2 + (y_1 - y'_1)^2 + (y_2 - y'_2)^2)^3} dy'_1 dy'_2 \\ &\sim \delta(W^+) (dW^+)^2 \frac{y_5^3}{(y_5^2 + (y_1 - y'_1)^2 + (y_2 - y'_2)^2)^3} dy'_1 dy'_2 \\ &\sim \delta(W^+) (dW^+)^2 \frac{(Z^0 + Z^3)}{(W \cdot Z)^3} dZ_1 dZ_2 \end{aligned} \quad (3.8)$$

where in the last line we have represented the boundary coordinates using (2.4), with $Z^+ = 0$, in order to get the answer in a form that will make it easy to make coordinate changes.

For example, if we wish to express the result in the hyperbolic coordinates (3.5) all we need to do is to express W and Z in terms of such coordinates. Let us define $\vec{n} = (n_1, n_2, n_3)$ to be a unit vector on a two sphere. On the surface $W^+ = 0$ we can parametrize $W^0 = \cosh \zeta$, $W_i = \sinh \zeta n_i$. It is then natural to take boundary coordinates $Z^0 = 1$, $Z^i = n^i$. We take the limit $\zeta \rightarrow \infty$ and we then have that

$$W^0 + W^3 \sim e^\zeta (1 + n_3) \sim e^\zeta (Z^0 + Z^3), \quad y'_{1,2} = \frac{Z_{1,2}}{Z^0 + Z^3} = \frac{n'_{1,2}}{(1 + n'_3)} \quad (3.9)$$

The last equation is simply the change of coordinates (2.10). We then find the following expression for the generators

$$\begin{aligned} E &\longrightarrow h_{MN}^E dX^N dX^M \sim \delta(W^+) (dW^+)^2 \frac{1}{(W^0 - W_i n_i)^3} d\Omega'_2 \\ P^i &\longrightarrow h_{MN}^{P^i} dX^N dX^M \sim \delta(W^+) (dW^+)^2 n'_i \frac{1}{(W^0 - W_i n'_i)^3} d\Omega'_2 \\ \mathcal{E}(\vec{n}') &\longrightarrow h_{MN}^{\mathcal{E}(\vec{n}')} dX^N dX^M \sim \delta(W^+) (dW^+)^2 \frac{1}{(W^0 - W_i n'_i)^3} \end{aligned} \quad (3.10)$$

In summary, we have computed the metric fluctuation that corresponds to the integrated insertion of the stress tensor that measures the energy deposited in the idealized calorimeters that we are placing at infinity at some position \vec{n}' on the two-sphere. In the original AdS coordinates the corresponding insertion would be localized on the horizon of AdS in Poincare coordinates ($z = \infty$). We found it convenient to express the results in a

couple of different coordinate systems that are regular at $z = \infty$ in order avoid having to take a limit. In these other coordinates we see that we are performing a measurement on the $W^+ = 0$ surface. This amounts to sampling the wavefunction of the particles in the bulk at $W^+ = 0$. At $W^+ = 0$ we have an H^3 subspace plus the null direction parametrized by W^- . The boundary of H^3 corresponds to the two sphere at infinity where we place the calorimeters.

The form of the equations (3.10) does not make explicit the Lorentz covariance of the expressions. In order to see this explicitly we can rewrite them as

$$P^\nu \rightarrow h_{MN}^{P\nu} dX^N dX^M \sim \delta(W^+) (dW^+)^2 \frac{Z^\nu}{(-W \cdot Z)^3} \frac{dS_\lambda^0 Z^\lambda}{Z^0} \quad (3.11)$$

where we sit at $Z^+ = 0$ and we are integrating over a spacelike surface inside $\sum_{\mu=0}^3 Z_\mu Z^\mu = 0$. The integration surface differential is defined to be such that $dS_\lambda^\nu Z^\lambda$ is parallel to Z^ν . Therefore the transformation of $dS_\lambda^0 Z^\lambda$ cancels the transformation of Z^0 .¹³

In the same way that we have discussed the graviton associated to energy flux measurements we can also consider the U(1) gauge field configurations associated to charge flow measurements on the boundary theory. The operator that corresponds to putting a counter at infinity at some specific location and measuring the charge corresponds to the following bulk gauge field configuration

$$\mathcal{Q}(\vec{n}') \rightarrow A_M dx^M \sim dW^+ \delta(W^+) \frac{1}{(W^0 - W^i n'_i)^2} \quad (3.12)$$

Having discussed the properties of the probe gravitons or gauge fields that represent our measurement, let us now turn to the field in the bulk that describes the state that we insert with the operator \mathcal{O} . We can think of a scalar source to make things simpler, although our results will be quite general. We are interested in obtaining the field configuration, ϕ , in the bulk of AdS_5 created by the insertion of the operator \mathcal{O} of dimension Δ . If we insert the operator $\int d^4 x' \phi_0(x') \mathcal{O}(x')$ on the boundary theory, then the bulk field configuration is given by [16, 15]

$$\begin{aligned} \phi(x, z) &= \int d^4 x' \phi_0(x') \frac{z^\Delta}{[(x - x')^2 + z^2]^\Delta} = \lim_{z' \rightarrow 0} \int d^4 x' \phi_0(x') \frac{1}{(z')^\Delta (W \cdot W')^\Delta} \\ \phi_q(W^+ = 0, W^-, W^\mu) &= \int d^4 x \frac{e^{iq \cdot x'}}{\left[-\frac{W^-}{2} - W^0 x'^0 + W^i x'^i - i\epsilon\right]^\Delta} \end{aligned} \quad (3.13)$$

where we have first rewritten the result in a way that allow us to easily change coordinates. In the last line we wrote the expression for the bulk field at $W^+ = 0$ in the case that $\phi_0(x') = e^{iq \cdot x'}$. Notice that we only need the wavefunction at $W^+ = 0$ since that is where the graviton perturbation is localized. It is not hard to do the integral in (3.13) explicitly. There is, however, a very simple way to see what the answer should be. We are creating a state that is a momentum eigenstate. For the moment let us set $q^\mu = (q^0, \vec{0})$.

¹³This is completely analog to the fact in classical electrodynamics that the power radiated by an accelerating charge is a Lorentz scalar.

The momentum generator corresponds to a bulk isometry generated by a Killing vector that becomes simpler at $W^+ = 0$,

$$P_x^\mu|_{W^+=0} = -2iW^\mu\partial_{W^-} \quad (3.14)$$

Of course, this is similar to the corresponding boundary statement (2.5). A wave function that diagonalizes all four of these operators has to be a plane wave in W^- and should be localized in the W^μ coordinates. In other words we have

$$\phi_q(W^+ = 0, W^-, W^\mu) \sim (q^0)^{\Delta-4} e^{iq^0W^-/2} \delta^3(\vec{W}) \quad (3.15)$$

where \vec{W} refers to a parametrization of the hyperboloid given by W^i with $i = 1, 2, 3$. In these coordinates, W^0 is just a function of \vec{W} since $W_\mu W^\mu = -1$ at $W^+ = 0$. $(q^0)^{\Delta-4}$ is a normalization constant that can be obtained from (3.13) by considering the dilatation operator acting on both sides. In other words, $\phi_0(x')$ is dimensionless so that ϕ scales as $(q^0)^{\Delta-4}$. The overall constant in (3.15) cancels out when we compute energy correlations. Note that in the y -AdS coordinates (3.2) the wavefunction with definite x -momentum, (3.15), is localized at $y_1 = y_2 = 0$, $y_5 = 1$ when $y^+ = 0$.

Therefore, the general result is that an incoming plane wave (with no spatial momentum) gives us a very peculiar wavefunction that is δ function localized in the H_3 subspace at the origin $W^i = 0$. In addition we find that the momentum in the W^- direction is proportional to the original energy. The wavefunction for an external operator with a generic value of the momentum q^μ can be obtained by performing a boost of this solution. The end result is again a wavefunction that is localized at a point in H_3 . It is localized at $\frac{\vec{W}}{W^0} = \frac{\vec{q}}{q^0}$. The momentum in the W^- direction is now $-\frac{\sqrt{-q_\mu q^\mu}}{2}$. The wavefunctions corresponding to plane waves have a divergent norm since a plane wave wavefunction has a divergent norm. One can consider the regularized external wavefunction in (2.28). In that case we find a finite norm. We discuss this case in more detail in appendix B. As expected, one finds that the delta function is smeared over a region $|\vec{W}| \sim \frac{1}{\sigma q}$. We will continue to discuss wavefunctions for plane waves, but having in mind that we will eventually smear the δ function in (3.15), as in (2.28).

Once we have the bulk wavefunction we can compute the energy flux one point function. By considering the effects of the metric perturbation (3.10), and considering an operator that creates the bulk wavefunction ϕ , we obtain the expression

$$\begin{aligned} \langle \mathcal{E}(\vec{n}') \rangle &= N^{-2} \int dW^- d\Sigma_3 \frac{1}{4\pi(W^0 - \vec{W} \cdot \vec{n}')^3} [(2i\partial_{W^-} \phi^*)(-2i\partial_{W^-} \phi)]|_{W^+=0} \\ N^2 &= \int dW^- \int d\Sigma_3 [\phi^*(-i\partial_{W^-} \phi) + c.c.]|_{W^+=0} \end{aligned} \quad (3.16)$$

where Σ_3 denotes the integral over the three dimensional hyperbolic space parametrized by W^μ , with $W_\mu W^\mu = -1$. The last factor, N^2 , is simply the total production cross section and it is related to the two point function of the operator insertion. In other words, the two point function $\langle 0|\mathcal{O}_q^\dagger \mathcal{O}_q|0\rangle = \|\mathcal{O}_q|0\rangle\|^2$ is the norm of the state. This norm is given in the bulk by the expression for N^2 . When we insert the wavefunction (3.15) we see that a single point in the integral over hyperbolic space contributes. We finally get the expected result $\langle \mathcal{E} \rangle = \frac{q}{4\pi}$, (2.29).

3.2 Energy flux one point functions in theories with gravity duals

Using the results above we are ready to calculate the energy one point functions for different type of sources. For a scalar source the symmetries imply the result (2.29). Because there are no free parameters we know this is the correct result and we did not need to go through the previous discussion. The situation is more interesting for current sources. In a theory that has a gravity dual the three point function of two currents and a stress tensor can be computed from the bulk interaction between two bulk photons and a bulk graviton that follows from the bulk Maxwell action

$$S = -\frac{1}{4g^2} \int d^5x \sqrt{g} F^2 \quad (3.17)$$

where g is the bulk gauge coupling. This term in the action also determines the two point function. Thus, we can see that the three point function will be determined and we will get a particular value for a_2 in (2.30). We can find this value by noticing that for $\mathcal{N} = 4$ Super Yang Mills we had $a_2 = 0$. Thus, any theory that has a gravity dual gives us $a_2 = 0$, as long as the two derivative approximation (3.17) is valid. In general there will be higher derivative corrections to this action. Up to field redefinitions there is a unique higher order operator that can contribute to the three point function

$$S = -\frac{1}{4g^2} \int d^5x \sqrt{g} F^2 + \frac{\alpha_1}{g^2 M_*^2} \int d^5x \sqrt{g} W^{\mu\nu\delta\rho} F_{\mu\nu} F_{\delta\rho} \quad (3.18)$$

where $W^{\mu\nu\delta\rho}$ is the Weyl tensor. Here M_* is some mass scale in the gravity theory determining the strength of the correction relative to the strength of the Maxwell term in the action. In order to see that this is the only operator that contributes we consider the possible three point vertices between two photons and a graviton in flat space. It turns out that there are only two possible structures. This onshell vertex is so constrained because there is no kinematic invariant that we can make purely with the external momenta, which all square to zero. In fact the two possible interaction vertices consistent with gauge invariance are

$$v_1 = \epsilon_{\mu\nu} [\epsilon_1^\mu k_2^\nu (\epsilon_2 \cdot k_1) + (1 \leftrightarrow 2) - k_1^\mu k_2^\nu (\epsilon_1 \cdot \epsilon_2)] \quad v_2 = \epsilon_{\mu\nu} k_1^\mu k_2^\nu (\epsilon_1 \cdot k_2) (\epsilon_2 \cdot k_1) \quad (3.19)$$

where $\epsilon_{1,2}^\mu$ are the polarization vectors of the gauge bosons and $\epsilon_{\mu\nu}$ the polarization vector of the graviton. They are all transverse $\epsilon_1 \cdot k_1 = 0$ and $\epsilon_{\mu\mu} = 0$. The first arises from the quadratic action (3.17) and the second from the higher order correction in (3.18).

We expect that the higher derivative corrections give us a deviation from a perfectly spherical energy distribution for the state created by the currents. Notice that the higher order correction will not contribute to the angle independent term in the energy one point function. The reason is that this term is related by the Ward identities to the two point function and the two point function is not corrected by the presence of the higher order operator in (3.18) because the Weyl tensor vanishes in AdS. A detailed computation in appendix D shows that the higher order term indeed contributes to the anisotropic contribution to the energy correlation function

$$a_2 = -\frac{48\alpha_1}{R_{\text{AdS}}^2 M_*^2} \quad (3.20)$$

Notice, in particular, that in non-supersymmetric weakly coupled *QCD* we expect that the higher derivative corrections are comparable to the radius of AdS since a_2 is of order one for weak coupling (2.32).

This anisotropy is intimately related to the anisotropy in the gravitational field that is produced by a fast moving photon. Let us consider a photon with high momentum $|p_-| \gg 1$. We focus on the problem in flat space for the moment. Such a fast moving particle produces a metric of the form

$$ds^2 = -dx^+ dx^- + d\vec{x}^2 + \delta(x^-)(dx^-)^2 h(\vec{x}), \quad \nabla^2 h = 0 \quad (3.21)$$

where ∇^2 is the flat laplacian in the transverse directions. This metric is an exact solution of Einstein's equations (with zero cosmological constant) and arbitrary higher derivative corrections [61]. The particular form of the solution for h depends on the coupling of the photons to the graviton. For the lowest order action (3.17) we find that $h_0 \sim \frac{p_-}{|x|}$ which is independent of the spin of the photon. Here we are focusing on the five dimensional case, so that we have three transverse directions \vec{x} . On the other hand, the second interaction (3.18) gives a function h of the form

$$h_1 \sim \frac{p_-}{M_*^2} \epsilon_i^* \epsilon_j \partial_i \partial_j \frac{1}{|x|} = \frac{3p_-}{M_*^2} \frac{|n_i \epsilon_i|^2 - \frac{1}{3} |\epsilon|^2}{|x|^3} \quad (3.22)$$

where $n_i = x_i/|x|$. Note that this contribution to the gravitational field is sensitive to the spin of the photon and it has a quadrupole form. In the case that we have a large number of photons, this quadrupole tensor would be proportional to the polarization density matrix of the photons.

Even though we've discussed the case of a photon, all that we have said so far can be extended to the case that we have a non-abelian gauge theory in the bulk, which corresponds to a non-abelian global symmetry in the boundary theory.

We have a similar story in the case that the inserted external operator is the stress tensor itself. Then there are three possible vertices and three parameters specifying the stress tensor three point function. One of these parameters is fixed by the Ward identities and it multiplies the three point function that we expect from the gravity action. The other two parameters multiply higher order gravity corrections. In fact, the three possible gravity vertices in five dimensions are

$$\begin{aligned} v_1 &= k_2^\mu \epsilon_{\mu\nu}^1 \epsilon^{\nu\delta} \epsilon^{\rho\delta} k_2^\rho + \frac{1}{4} \epsilon_{\mu\nu}^1 \epsilon^{2\mu\nu} \epsilon^3_{\delta\rho} k_1^\delta k_2^\rho + \text{cyclic} \\ v_2 &= (k_3^\mu \epsilon_{\mu\nu}^1 \epsilon^{\nu\delta} k_3^\delta) (\epsilon^3_{\rho\sigma} k_1^\rho k_2^\sigma) + \text{cyclic} \\ v_3 &= (\epsilon_{\mu\nu}^1 k_2^\mu k_2^\nu) (\epsilon^2_{\delta\sigma} k_3^\delta k_3^\sigma) (\epsilon^3_{\rho\gamma} k_1^\rho k_1^\gamma) \end{aligned} \quad (3.23)$$

Such vertices arise from terms in the action of the form

$$S = \frac{M_{pl}^3}{2} \left[\int d^5x \sqrt{g} R + \frac{\gamma_1}{M_{pl}^2} W_{\mu\nu\delta\sigma} W^{\mu\nu\delta\sigma} + \frac{\gamma_2}{M_{pl}^4} W_{\mu\nu\delta\sigma} W^{\delta\sigma\rho\gamma} W_{\rho\gamma}{}^{\mu\nu} \right] \quad (3.24)$$

This is one way to parametrize the higher derivative corrections. In principle we can have another curvature cubed term but it does not contribute to the three point function [62].

In fact, in [62] such corrections were computed for various string theories. They found that γ_1 and γ_2 are non-zero in the bosonic string, only γ_1 is nonzero in the heterotic string and both $\gamma_1 = \gamma_2 = 0$ in the type II superstrings. Incidentally, $\frac{1}{N}$ corrections to this action, yielding an R^4 term, were computed for type IIB superstrings in $AdS_5 \times S^5$ in [63] and their effect on the 3 point function of stress energy tensors in $\mathcal{N} = 4$ SYM was discussed.

One can compute the contributions of the higher order terms in the action to t_2 and t_4 as defined in (2.37). After performing some calculation described in appendix D, we find

$$\begin{aligned} t_2 &= \frac{48\gamma_1}{R_{\text{AdS}}^2 M_{\text{pl}}^2} + o\left(\frac{\gamma_2}{R_{\text{AdS}}^4 M_{\text{pl}}^4}\right) \\ t_4 &= \frac{4320\gamma_2}{R_{\text{AdS}}^4 M_{\text{pl}}^4} \end{aligned} \quad (3.25)$$

to leading order in the γ_i . In addition we have assumed that the contribution of the W^3 operator to t_2 will be smaller than the one from the W^2 operator. This is expected in the large radius limit because W^3 has more derivatives. For an $\mathcal{N} = 1$ supersymmetric theory $t_4 = 0$. Using (3.25) and (2.39) we get the expression for the R^2 coefficient that was derived in [64, 51].

Notice that the presence of the first correction to the action (3.24) can also change the angular independent part of the energy flux one point function. This change should be compensated precisely by a change in the stress tensor two point function in order to obey the Ward identity.

We could also consider charge one point functions. In that case there are two (parity preserving) structures for the three point function [42, 43, 65]. The coefficient of one of them is determined by the Ward identities and arises only when we have a non-abelian gauge symmetry in the bulk. It comes from the usual bulk term of the form $\int Tr[F^2]$. The second structure arises from a bulk term of the form $\int Tr[F_{\mu\nu} F^{\nu\delta} F_\delta{}^\mu]$, or more generally, from a bulk coupling of the form $\int f_{abc} F_{\mu\nu}^a F^{b\nu\delta} F_\delta{}^c{}^\mu$ with a totally antisymmetric f_{abc} . Notice that these terms do not necessarily come from a non-abelian gauge symmetry. They could come from a coupling between three different U(1) gauge field strengths in the bulk.

The parity odd terms come from Chern Simons couplings in five dimensions [16, 65]. For example, for a gauge field we can have $\int A \wedge F \wedge F$ or its non-abelian generalization.

3.3 Comments on the n point functions

After this discussion on one point functions, let us move on to n point functions. All we need to do is to consider metric fluctuations which contain several insertions of the energy flux operator. In general we would have to worry about the bulk tree level interactions among the bulk gravitons corresponding to the insertions of the operators. In our case, there is an important simplification. This is due to the fact that the following plane wave solutions¹⁴ are exact solutions of Einstein's bulk equations [67, 68]

$$ds^2 = ds_{\text{AdS}_5}^2 + (dW^+)^2 \delta(W^+) h(w) \quad (3.26)$$

¹⁴For applications of this type of solutions in confining backgrounds see [66].

where $h(w)$ is a function defined on the transverse space, which in this case is a hyperbolic space H_3 of radius one, given by $-(W^0)^2 + (W^1)^2 + (W^2)^2 + (W^3)^2 = -1$. The function $h(w)$ obeys the Laplace equation on this hyperbolic space

$$\vec{\nabla}_w^2 h = 3h \quad (3.27)$$

Of course, one can check that¹⁵

$$h_{\vec{n}'} = \frac{2i}{4\pi(W^0 - W^i n'_i)^3} \quad (3.28)$$

is a solution and so will be an arbitrary superposition

$$h = \sum_{j=1}^n h_{\vec{n}'_j} \quad (3.29)$$

which represents the insertion of n calorimeters at angular positions given by \vec{n}'_j , $j = 1, \dots, n$. This is summing all the gravity tree diagrams. We should now consider the propagation of the wavefunction on the background of this plane wave geometry. We want to consider the effects of each $h_{\vec{n}'_j}$ to first order in its strength but the combined effect of all of them. Let us recall how we would analyze this problem in flat space first. We consider a flat space plane wave of the form

$$ds^2 = -dx^+ dx^- + (dx^+)^2 f(x^+) h(\vec{x}) + d\vec{x}^2 \quad (3.30)$$

and we will eventually take the limit where $f(x^+) \rightarrow \delta(x^+)$. The scalar field obeys the equation

$$-4\partial_- \partial_+ \phi - 4f(x^+) h(\vec{x}) \partial_-^2 \phi + \vec{\nabla}^2 \phi - m^2 \phi = 0 \quad (3.31)$$

We now assume that $f(x^+)$ is nonzero only within some small neighborhood of the origin $-\epsilon < x^+ < \epsilon$. This implies that $f(x^+)$ varies rapidly. Thus we assume that this rate of variation is much faster than the rate of variation of the wavefunction along the rest of the coordinates. In that region we can then approximately solve the wave equation (3.31) as

$$\phi(x^+ = \epsilon) = e^{-\int_{-\epsilon}^{\epsilon} f(x^+) h \partial_-} \phi(x^+ = -\epsilon) \rightarrow \phi(x^+ = \epsilon) = e^{-h \partial_-} \phi(x^+ = -\epsilon) \quad (3.32)$$

Generalizing this method to our case of interest we find that

$$\phi(W^+ = \epsilon, W^-, W^\mu) = e^{-h \partial_{W^-}} \phi(W^+ = -\epsilon, W^-, W^\mu) \quad (3.33)$$

where W^μ denotes a point in H_3 . This is nothing else than a translation of magnitude h in the W^- direction. The same type of behavior was observed for scattering of particles off shock waves in [69] and was used to study four point functions in the context of the AdS/CFT correspondence in [79].

The computation we want to do involves the overlap of the final state with the initial state in the background deformed by the insertion of the plane wave. In addition, we need

¹⁵Here the normalization factor is fixed such that we obtain the total energy upon integration.

to divide by the norm (3.16). If we write $h = \sum_j h_{\vec{n}'_j}$ and we expand in each of the $h_{\vec{n}'_j}$ to first order we get the n point function

$$\langle \prod_j \mathcal{E}(\vec{n}'_j) \rangle = N^{-2} \int_{W^+=0} dW^- d\Sigma_3 \left[(i\partial_{W^-} \phi^*) \prod_{j=1}^n h_{\vec{n}'_j}(W) [(-\partial_{W^-})^n \phi] + c.c. \right] \quad (3.34)$$

where the integral over $d\Sigma_3$ is over the hyperboloid $W^\mu W_\mu = -1$, $\mu = 0, 1, 2, 3$. Let us specialize this expression to the case that we have a plane wave external state, which leads to (3.15). In that case we find that all the $h_{\vec{n}'_j}$ are evaluated at $\vec{W} = 0$ so that they become independent of the angle. Thus we get that not only the one point function is uniform but also all the n point functions are uniform as well. This implies that there are no fluctuations in the energy and an observer would see a uniform energy deposition in all the detectors. In other words, we get

$$\langle \mathcal{E}(\vec{n}'_1) \cdots \mathcal{E}(\vec{n}'_n) \rangle = \left(\frac{q}{4\pi} \right)^n \quad (3.35)$$

This is what we would expect by thinking that fragmentation is very rapid at strong coupling as suggested in [2, 4]. For a state with a generic, but definite, momentum we find

$$\mathcal{E}_{q^\mu}(\vec{n}') = \frac{1}{4\pi} \frac{(q^2)^2}{(q^0 - \vec{q} \cdot \vec{n}')^3} \quad (3.36)$$

which is simply the boosted version of the uniform distribution $\mathcal{E} = \frac{q^0}{4\pi}$ that we get for the case where $\vec{q} = 0$.

This is the result for plane wave states. If one considers a generic state, then there can be fluctuations, but such fluctuations are parametrized by the fact that we have a wavefunction for momentum. In other words, one can write the wavefunction $\phi_0(x)$ appearing in (3.13) in momentum space as $\tilde{\phi}_0(p) \equiv \int d^4x e^{-ip \cdot x} \phi_0(x)$. We consider only wavefunctions which are nonvanishing in the forward light-cone $p^2 < 0$, $p^0 > 0$. We could consider other wavefunctions but the corresponding operators vanish when they act on the vacuum and will not contribute. Thus, in the formulas below we imagine that p^μ is restricted to be in the forward light-cone. Then we can write the bulk wavefunction as

$$\phi(W^+ = 0, W^-, W^\mu) = \int_0^\infty \frac{d\lambda}{\lambda} \lambda^\Delta e^{i\lambda W^- / 2} \tilde{\phi}_0(\lambda W^\mu) \quad (3.37)$$

We see that for a plane wave with purely timelike momentum we should set $\tilde{\phi}_0 = \delta(p^0 - q^0) \delta^3(\vec{p})$ and we recover (3.15). Inserting (3.37) into (3.34) we obtain

$$\begin{aligned} \langle \prod_{i=1}^n \mathcal{E}(\vec{n}'_i) \rangle &= N^{-2} \int d^4p \rho(p) \prod_{i=1}^n \frac{p^4}{4\pi(p^0 - \vec{p} \cdot \vec{n}'_i)^3} \rho(p) = N^{-2} (p^2)^{\Delta-2} |\tilde{\phi}_0(p)|^2, \\ N^2 &= \int d^4p (p^2)^{\Delta-2} |\tilde{\phi}_0(p)|^2 \end{aligned} \quad (3.38)$$

The factor of $(p^2)^{\Delta-2}$ appears when we consider the norm of a state that has momentum p , see appendix A. This factor is determined by the dilatation operator. In appendix B we compute ρ for the localized wavefunction $\phi_0(x)$ given in (2.28).

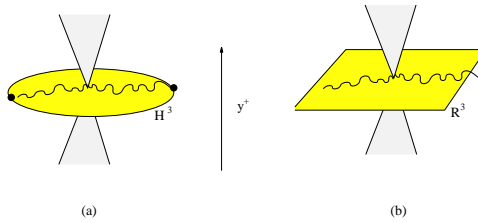


Figure 4: (a) The AdS computation of the energy correlators involves gravitons that propagate from the boundary to the interior on an H_3 subspace of the full AdS_5 space. The gravitons originate on the boundary of H_3 , at the point where the calorimeter is inserted, and propagate on H^3 to the interior. (b) Since the falling string state is localized on H_3 we can approximate the computation by a flat space one.

The final picture is that for a generic operator insertion we have a superposition of the results for each momentum, given by (3.36) with a probability weight given by $\rho(p)$ which is giving us the probability of exciting the mode with momentum p^μ in the conformal field theory (or in the bulk gravity theory).

For a generic $\tilde{\phi}_0(p)$ (3.38) gives non-trivial functions of the angles. The final picture for what we would see in each event is actually very simple. After we measure the energy on four of the calorimeters in each event, we can determine the value of p that is contributing and, therefore, the energies in all other calorimeters is determined. See appendix B for a longer discussion of this point. In other words, from event to event, we have some random variations which are completely captured by the distribution of momenta $\rho(p)$. In the bulk picture, we have a pointlike particle in the bulk with some wavefunction. Measuring all the energies is tantamount to measuring the position of the particle on H_3 and its momentum in the direction W^- when it crosses $W^+ = 0$. We can view this as the horizon of AdS. We can say that we are simply measuring the momentum of the particle as it crosses the AdS_5 horizon. In the approximation that we have a pointlike particle we have a small number of random (quantum) variables characterizing the event. We only have the position or momentum of the particle when it crosses the horizon. When we consider a string we have an infinite number of degrees of freedom and we can have much more variation in the energy deposition patterns.

4. Stringy corrections

In this section we study stringy corrections to the gravity results. First we consider a flat space problem that is closely related to the problem we need to solve in AdS. We then use these results to compute the leading order stringy corrections to the gravity results. Finally we study the small angle behavior of the two point function and we find the stringy version of the operator product expansion we discussed above.

4.1 Strings probed by plane waves

Let us first make the approximation that the AdS space is weakly curved and let us approximate the problem as that of strings in flat space, see 4. In fact, we have seen that the state

created by the operator insertion is localized on the transverse surface, the H_3 subspace. In addition, the energy flux operator corresponds to a graviton localized at $W^+ = 0$. Thus, we can just look at the problem in a neighborhood of this point and approximate it as a flat space problem where we have a particle, or a string, with nonzero p_- which it is being probed by gravitons with $p_- = 0$ that are localized along y^+ . Note that the probe gravitons are extended in the y^- direction.

More explicitly, we can consider a flat space problem where we have a string with a non-zero value of p_- which crosses a gravitational plane wave of the form

$$ds^2 = -dy^+ dy^- + (dy^+)^2 \delta(y^+) h + d\vec{y}^2 \quad (4.1)$$

where h is a function of the transverse coordinates obeying $\vec{\nabla}^2 h = 0$. Due to the symmetries of the problem it is convenient to choose light cone gauge where $y^+ = -2\alpha' p_- \tau$. Recall that $p_- < 0$ is the momentum conjugate to y^- . Following the usual steps that lead to light cone quantization we find that we get the following light cone gauge Lagrangian for the transverse dimensions

$$S = \frac{1}{4\pi\alpha'} \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma d\tau [(\partial_\tau \vec{y})^2 - (\partial_\sigma \vec{y})^2] - \frac{1}{2\pi} p_- \int_0^{2\pi} d\sigma h(\vec{y}(\tau=0, \sigma)) \quad (4.2)$$

Notice that the h dependent term is localized at $\tau = 0$, at a single value of the worldsheet time. Thus, the string propagates freely in flat space away from $y^+ = 0$, or $\tau = 0$. We will also assume that near $y^+ \sim 0$ the string is localized near $\vec{y} = 0$ in the transverse directions. We can then compute correlation functions from the expression

$$\langle \Psi | e^{-ip_- \int_0^{2\pi} \frac{d\sigma}{2\pi} h(\vec{y}(\sigma))} |_{\tau=0} | \Psi \rangle \quad (4.3)$$

where $|\Psi\rangle$ the full wavefunction of the string state in the light cone gauge theory at $\tau = 0$. We consider a function h which is a sum of a finite number of plane waves, $h = \sum_j h_j e^{i\vec{k}_j \vec{y}}$, and we expand to linear order in each perturbation h_j . The mass shell condition is $\vec{k}_j^2 = 0$. This implies that \vec{k} is complex. (When we go back to the AdS problem it will be natural to take the component of k along the radial direction to be purely imaginary and the others to be real.) Let us assume that the string state corresponds to the ground state for the bosonic oscillators excitations, at least for the bosonic transverse directions where the momenta \vec{k} are nonzero. We then find that we have to compute correlation functions of the form

$$\begin{aligned} & (-ip_-)^n \langle \psi_{\text{cm}} | \prod_j e^{i\vec{k}_j \vec{y}} | \psi_{\text{cm}} \rangle \langle 0 | \prod_j \int \frac{d\sigma_j}{2\pi} e^{i\vec{k}_j \vec{y}_{\text{osc}}(\sigma)} | 0 \rangle \\ & \sim (-ip_-)^n \langle \psi_{\text{cm}} | \prod_j e^{i\vec{k}_j \vec{y}} | \psi_{\text{cm}} \rangle \prod_j \int \frac{d\sigma_j}{2\pi} \prod_{j<i} |2 \sin \frac{\sigma_i - \sigma_j}{2}|^{\alpha' \vec{k}_i \cdot \vec{k}_j} \end{aligned} \quad (4.4)$$

where we have separated out the contribution from the center of mass and the oscillators. Since the center of mass wavefunction is well localized we expect no contribution from it. Namely, we imagine a wavefunction which is localized near $\vec{y} = 0$ and thus we simply need to evaluate the plane waves in (4.4) at zero which just gives one. Namely,

$\langle \psi_{\text{cm}} | \prod_j e^{i\vec{k}_j \vec{y}} | \psi_{\text{cm}} \rangle \sim 1$. Note that if we neglect the oscillator contributions we recover the gravity result following from (3.33). Therefore, the nontrivial contribution comes from the oscillators. Notice that these integrals are convergent if the k 's are all small enough. In the case of the two point function we have

$$\int_0^{2\pi} \frac{d\sigma}{2\pi} \left| 2 \sin \frac{\sigma}{2} \right|^{\alpha' k_1 \cdot k_2} = \frac{2^{\alpha' k_1 \cdot k_2}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{2} + \frac{\alpha' k_1 \cdot k_2}{2}\right)}{\Gamma\left(1 + \frac{\alpha' k_1 \cdot k_2}{2}\right)} = 1 + \frac{\pi^2}{24} (\alpha' k_1 \cdot k_2)^2 + \dots \quad (4.5)$$

In the second line, the 1 corresponds to the gravity result and the second term is the first correction. Naively, one might have expected the first correction to be of order α' . However, the first term vanishes and the order α'^2 term is the first non vanishing one.

It is convenient to rewrite this result in position space. We find that the gravity result plus the leading order correction can be written as

$$N^{-2} \int_{y^+=0} dy^- d^3 y \left[i\phi^* \partial_{y^-}^3 \phi + c.c. \right] \left[h_1(y) h_2(y) + \frac{\pi^2}{24} \alpha'^2 (\partial_i \partial_j h_1(y) \partial_i \partial_j h_2(y)) \right],$$

where
$$N^2 = \int_{y^+=0} dy^- d^3 y \left[i\phi^* \partial_{y^-} \phi + c.c. \right] \quad (4.6)$$

where ϕ is the wavefunction of the center of mass of the closed string state and h_1 and h_2 are two graviton plane wave states. We have also normalized the result.

In fact, from our discussion we can easily see the origin of the α' corrections to the three graviton vertex in various string theories. These were computed in [62]. We first consider the case where there is just one probe graviton in (4.4). α' corrections can only arise if the initial state contains bosonic oscillators in the transverse directions. For a graviton in the superstring we have no bosonic oscillators in the initial state, only fermion zero modes. Thus the vertex is the same as the gravity one. In the case of the heterotic string the graviton contains fermion zero modes for the right movers and a bosonic oscillator for the left movers. Such an oscillator can give rise to momentum dependent terms of the form given by the second vertex in (3.23), but not like the third in (3.23). Finally, in the case of the bosonic string a graviton with indices in the transverse directions involves bosonic oscillators for both left and right movers and gives rise to a vertex like the third in (3.23) (plus the first two, of course).

It is interesting that the string result is finite. One might have worried that since we are using $\delta(y^+)$ wavefunctions we would obtain divergencies. As we will see in more detail below, the Regge behavior of the scattering amplitudes in string theory ensures that the results are finite.

4.2 Leading order α' corrections to the two point function

We now generalize this result to curved space. We can simply replace the ordinary derivatives in (4.6) by covariant derivatives. However, in the AdS context h obeys an equation of the form $\nabla^2 h = 3h$, or more precisely $\nabla^2 h = \frac{3}{R_{\text{AdS}}^2} h$, so that terms that would have been zero in flat space are non-zero in AdS, so we seem to be faced with an ambiguity. However, these ambiguities only affect terms that do not have angular dependence, at least for the

first correction. Thus, such terms only correct the constant part of the energy correlations. We can fix such corrections by demanding that we obey the energy conservation conditions. It is convenient to think about the problem in the hyperbolic coordinates (3.5). The graviton wavefunction associated to the insertion of a calorimeter at \vec{n}' on the two sphere is given by (3.16)

$$h \sim \frac{1}{(W^0 - \vec{W} \cdot \vec{n}')^3} \sim \frac{1}{\left(1 + \frac{|\vec{W}|^2}{2} - \vec{W} \cdot \vec{n}'\right)^3} \quad (4.7)$$

where we have expanded the result around $\vec{W} \sim 0$. We have already seen that the center of mass wavefunction is localized near the origin of hyperbolic space if we have a state created by an operator with zero spatial momentum on $R^{1,3}$, see (3.15). Thus, we can evaluate the derivatives in (4.6) and then set $\vec{W} = 0$. This gives

$$\langle \mathcal{E}(\vec{n}'_1) \mathcal{E}(\vec{n}'_2) \rangle = \left(\frac{q^0}{4\pi}\right)^2 \left[1 + \frac{\pi^2}{24} \frac{\alpha'^2}{R_{\text{AdS}}^4} (\partial_i \partial_j h \partial_i \partial_j h)|_{\vec{W}=0} + \text{const} \right] \quad (4.8)$$

where the constant is an angle independent term that we cannot compute purely in flat space. It can be fixed so that we obey the energy conservation condition. In the end we find

$$\langle \mathcal{E}(\vec{n}'_1) \mathcal{E}(\vec{n}'_2) \rangle = \left(\frac{q^0}{4\pi}\right)^2 \left[1 + \frac{6\pi^2}{\lambda} \left(\cos^2 \theta_{12} - \frac{1}{3} \right) + \dots \right] \quad (4.9)$$

for $\mathcal{N} = 4$ super Yang Mills. where $\cos \theta_{12} = \vec{n}'_1 \cdot \vec{n}'_2$. We see that, as expected here, the distribution rises in the forward and backward regions. We have fixed the constant term in the correction by demanding that the integral over one of the angles gives the total energy. The dots in (4.9) denote higher order terms in the $1/\sqrt{\lambda}$ expansion.

In this derivation we have assumed that the state we are considering has no oscillators excited along the three transverse AdS directions. In the case of the superstring we can still have a massless mode with indices in the transverse AdS directions since those can be accounted for by fermion zero modes on the string worldsheet in light cone gauge. The result (4.9) is very general and holds for any theory with a ten dimensional weakly coupled dual with an AdS_5 factor if we replace $1/\lambda \rightarrow \alpha'^2/R_{\text{AdS}}^4$, under the assumption that we are creating a ten dimensional massless closed string with the external operator.

4.3 Corrections to the n point function

We now consider the n point function $\langle \mathcal{E}(\vec{n}_1) \dots \mathcal{E}(\vec{n}_n) \rangle$. We have seen that the gravity result is just a constant. Let us compute the stringy corrections. The leading deviation can be computed by expanding the full expression (4.4) up to quadratic order in products of $k_i \cdot k_j$. The resulting correction is basically the same as the one contributing to the two point function (4.9). In order to see something new we can go to cubic order in the products $k_i \cdot k_j$. In the end this gives us a a correction to the n point function which looks like

$$\begin{aligned} \langle \mathcal{E}(\vec{n}_1) \dots \mathcal{E}(\vec{n}_n) \rangle &= \left(\frac{q}{4\pi}\right)^n \left[1 + \sum_{i < j} \frac{6\pi^2}{\lambda} \left[(\vec{n}_i \cdot \vec{n}_j)^2 - \frac{1}{3} \right] + \right. \\ &\quad \left. + \frac{\beta}{\lambda^{3/2}} \left[\sum_{i < j < k} (\vec{n}_i \cdot \vec{n}_j)(\vec{n}_j \cdot \vec{n}_k)(\vec{n}_i \cdot \vec{n}_k) + \dots \right] + o(\lambda^{-2}) \right] \end{aligned} \quad (4.10)$$

where β is a numerical coefficient¹⁶ and the dots denote terms that are necessary to ensure that the integral over each of the angles gives zero as well as a term that corrects the coefficient of the $(\vec{n}_i \vec{n}_j)^2$ term by an order $\lambda^{-3/2}$ amount.

Thus, we find that for a strongly coupled field theory the energy distribution is uniform with small fluctuations which have an amplitude of order $1/\sqrt{\lambda}$. In other words, $\delta\mathcal{E}/\mathcal{E} \sim \frac{1}{\sqrt{\lambda}}$. The two point function of these fluctuations is given by the first non-constant term in (4.10). One might have thought that these fluctuations would be gaussian. However, we find that the three point function of the fluctuations is of order $\lambda^{-3/2}$. Thus, when we normalize the two point functions to one, the three point functions are of order one. Thus we conclude that the fluctuations are not even approximately gaussian. More explicitly, we can define a fluctuation operator

$$\delta = \frac{\mathcal{E} - \langle \mathcal{E} \rangle}{\langle \mathcal{E} \rangle}, \quad \text{and} \quad \hat{\delta} = \sqrt{\lambda} \delta \quad (4.11)$$

where the operator $\hat{\delta}$ is defined so that its two point function is independent of λ . We then have

$$\begin{aligned} \langle \hat{\delta}(\vec{n}_1) \rangle &= 0 \\ \langle \hat{\delta}(\vec{n}_1) \hat{\delta}(\vec{n}_2) \rangle &= 6\pi^2 \left[(\vec{n}_1 \cdot \vec{n}_2)^2 - \frac{1}{3} \right] \left[1 + o(\lambda^{-1/2}) \right] \\ \langle \hat{\delta}(\vec{n}_1) \hat{\delta}(\vec{n}_2) \hat{\delta}(\vec{n}_3) \rangle &= \beta [(\vec{n}_1 \cdot \vec{n}_2)(\vec{n}_1 \cdot \vec{n}_3)(\vec{n}_2 \cdot \vec{n}_3) + \dots] \end{aligned} \quad (4.12)$$

We see that the three point function is not parameterically suppressed relative to the two point function. Of course, they are both suppressed relative to the gravity result.

4.4 Stringy corrections to charge two point functions

We now consider the two point function of a charge that is dual to a closed string mode. For example, we can pick one of the $\text{SO}(6)$ currents in $\mathcal{N} = 4$ super Yang Mills. More generally we consider a current associated to a symmetry that is carried by fields in the adjoint representation in the dual field theory. Imagine that the current comes from Kaluza Klein reduction. Then the corresponding vertex operator, in light cone gauge, has the form $A_+ \rightarrow \partial_\tau \varphi e^{ik \cdot x}$ where φ is one of the internal dimensions. We assume that the state we are measuring, $|\Psi\rangle$, does not have any oscillator excited in the φ direction and that it is not charged. Thus the one point function of the charge, is zero. The two point function is actually infinite in the gravity approximation. This is due to the Feynman diagram in 5 (b). This is intimately related to the fact that (3.12) is not an exact solution of the gravity equations, but it sources a gravitational plane wave proportional to F_{+i}^2 , which leads to the square of a $\delta(y^+)$ function. We did not run into this problem in the gravity theory because there are no diagrams of this form due to the fact that the gravitational shock wave is an exact solution of the theory. Even in the gravity theory we could have run into this problem if we had had a higher derivative contact interaction that brings together two gravitons, as in 5 (c).

¹⁶ $\beta = -1728 \int_0^{2\pi} \frac{d\sigma_1 d\sigma_2}{(2\pi)^2} \log[2 \sin \frac{\sigma_1}{2}] \log[2 \sin \frac{\sigma_2}{2}] \log[2 |\sin \frac{(\sigma_1 - \sigma_2)}{2}|] \sim 518 \pm 5.$

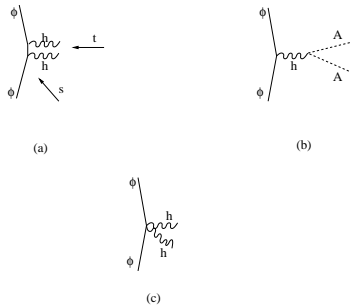


Figure 5: (a) Feynman diagrams that lead to energy correlation functions. The gravitons do not interact before they touch the falling state, ϕ . We have also indicated the t and s channels as we define them in the text. (b) Diagram that leads to a divergence in the charge correlation function. The intermediate state is a graviton and the AAh vertex comes from the Maxwell term in the action. This divergence is cured by going to string theory and exploiting the Regge behavior of the amplitudes. In (c) we draw a diagram that can arise due to a higher derivative contact interaction in the gravity theory which could lead to a divergence in the gravity approximation.

Let us now compute the two point function in string theory. The corresponding flat space expression is similar to (4.5), but with an extra factor coming from the contractions of the $\partial_\tau\varphi$ field coming from the two vertex operators. We get a result proportional to

$$\int_0^{2\pi} \frac{d\sigma}{(2\pi)} \left| 2 \sin \frac{\sigma}{2} \right|^{\alpha' k_1 \cdot k_2 - 2} = \frac{2^{\alpha' k_1 \cdot k_2 - 2}}{\sqrt{\pi}} \frac{\Gamma(-\frac{1}{2} + \frac{\alpha' k_1 \cdot k_2}{2})}{\Gamma(\frac{\alpha' k_1 \cdot k_2}{2})} \sim -\frac{\alpha' k_1 \cdot k_2}{4} + \dots \quad (4.13)$$

We have defined the integral by analytic continuation in $k_1 \cdot k_2$. We see that we get a perfectly finite answer in string theory. The string theory answer even goes to zero as we take the small momentum limit. Translating this flat space result (4.13) to AdS as we did above we get

$$\langle \mathcal{Q}(\vec{n}_1) \mathcal{Q}(\vec{n}_2) \rangle = \frac{\gamma}{\sqrt{\lambda}} \vec{n}_1 \cdot \vec{n}_2 = \frac{\gamma}{\sqrt{\lambda}} \cos \theta_{12} \quad (4.14)$$

where γ is a positive numerical coefficient. This result has the angular dependence that one would intuitively expect, with the two oppositely charged particles going in opposite directions.

Let us emphasize once more an important point. Due to the fact that we are considering shock waves which are highly localized, with $\delta(W^+)$ wavefunctions, it is important to perform the computation in string theory rather than first taking the low energy limit of string theory and then doing the computation. We will revisit this point later.

Another interesting situation arises when we consider currents that act only on fields in the fundamental representation, such as flavor symmetries. In this case the currents live on D-branes in the bulk. For simplicity let us assume that we have two D-branes with two different $U(1)$ gauge fields in the bulk. Let us call them $U(1)_A$ and $U(1)_B$. We could imagine a QCD-like theory where $U(1)_A$ and $U(1)_B$ are different flavor number symmetries. At leading order in N we detect these charges in the detector only if we create a mesonic operator that contains the corresponding quarks. Consider a situation where we have a

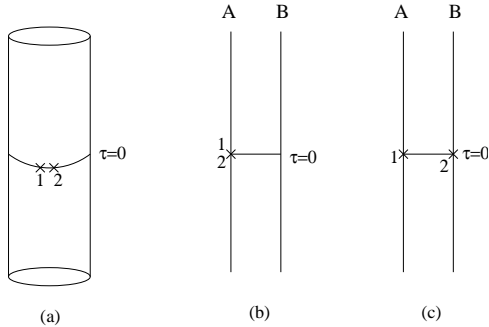


Figure 6: (a) Worldsheet vertex operator insertions for charge correlators associated to closed string gauge fields. These charges are carried by the bulk of the worldsheet. In (b) and (c) we consider charges carried by open strings. We consider an open string stretching between two different branes called A and B . In (b) we consider the charge two point function for the $U(1)_A$ living on the brane A . The two vertex operators are inserted at the same point. In (c) we consider the charge two point function for charge living on the brane A , $U(1)_A$, and the charge living on brane B , $U(1)_B$. The vertex operators are inserted on different boundaries and the result is non-singular.

lorentz scalar meson where the quark is charged under $U(1)_A$ and the anti-quark under $U(1)_B$. In such a situation we expect to find only one charge of each type in the detector. In this case the charge one point functions are $\langle \mathcal{Q}_A(\vec{n}) \rangle = \langle \mathcal{Q}_B(\vec{n}) \rangle = \frac{1}{4\pi}$. What is the charge two point function for $U(1)_A$? From the boundary field theory point of view we expect it to be zero for generic angles since the charged quark can be detected only at one particular angle, since we are working to leading order in N where we do not create quark anti-quark pairs. On the other hand, to leading order in N , from the gravity plus maxwell theory in the bulk we get a completely spherically symmetric distribution of charge. In this case, the divergent term coming from the Feynman diagram in 5 (b), is subleading in N and we do not consider it (string theory ought to make this $1/N$ correction finite too). In other words, the two point function for the charges is $\langle \mathcal{Q}_A(\vec{n}) \mathcal{Q}_A(\vec{n}') \rangle_{\text{gravity}} = \frac{1}{(4\pi)^2}$. This contradicts the field theory expectations. The resolution is that the stringy corrections are so large that they completely change the gravity result. Let us first see how this works in the flat space case. Here, we can quantize the open string in light cone gauge and we will get an action very similar to the one we had for the closed string except that the photon vertex operator, which is inserted at $\tau = 0$, is also inserted at $\sigma = 0$ at one of the boundaries of the open string. We find that

$$\langle \psi_{\text{cm}} | e^{i\vec{k}_1 \vec{y}} e^{i\vec{k}_2 \vec{y}} | \psi_{\text{cm}} \rangle \langle 0 | e^{i\vec{k}_1 \vec{y}_{\text{osc}}(0,0)} e^{i\vec{k}_2 \vec{y}_{\text{osc}}(0,0)} | 0 \rangle \quad (4.15)$$

If we ignore the oscillators we go back to the gravity result. However, the contribution from the oscillators involves a singularity, since both vertex operators are evaluated at the same point. Formally, this gives a contribution of the form $0^{2\alpha' k_1 \cdot k_2}$. If $k_1 \cdot k_2 > 0$, then we see that this vanishes. Thus, if we define the answer by analytic continuation we get zero for all values of $k_1 \cdot k_2$, including the physical values of $k_1 \cdot k_2$ for our problem (which are negative). The reason that stringy corrections have such a large effect is that we start with singular wavefunctions for the photon, which contain a $\delta(y^+)$. If we had started with

a smooth wavefunction in the x^+ dimension we would have integrated the vertex operators along the τ direction on the boundary of the open string worldsheet and we would have obtained a non-vanishing function of $k_1.k_2$.

On the other hand, if we compute the two point function for the two different U(1) charges, the charge carried by the quark and the charge carried by the antiquark, then we get the vertex operators at opposite points of the string and we obtain a finite answer

$$\langle \psi_{\text{cm}} | e^{i\vec{k}_1 \vec{y}} e^{i\vec{k}_2 \vec{y}} | \psi_{\text{cm}} \rangle \langle 0 | e^{i\vec{k}_1 \vec{y}_{\text{osc}}(0,0)} e^{i\vec{k}_2 \vec{y}_{\text{osc}}(0,\sigma=\pi)} | 0 \rangle \sim 2^{2\alpha' k_1.k_2} \quad (4.16)$$

In this case the leading order α' correction to the two point function reads

$$\langle \mathcal{Q}_A(\vec{n}_1) \mathcal{Q}_B(\vec{n}_2) \rangle = \frac{1}{(4\pi)^2} \left[1 - \frac{8 \log 2}{\sqrt{\lambda}} \cos \theta_{12} \right] \quad (4.17)$$

We see that there is a tendency for the two charges to go in opposite directions, as one naively expects. Of course, at weak coupling the quark and the antiquark fly in opposite directions (if we have the simplest operator which contains only a quark anti-quark pair). If we were to consider a higher point function we would get zero again.

The general lesson is that when we compute charge correlators it is very important to understand the effects of stringy corrections.

Once we consider finite N corrections we do not expect the two point functions for the same U(1) to be exactly zero.

4.5 Small angle behavior of the two point function and the operator product expansion

In this section we study the small angle behavior of the two point functions using string theory. The leading order correction to the energy flux two point function (4.9) is analytic at small angles, i.e. when $\theta_{12} \rightarrow 0$. As we will explain below this is no longer the case once all the α' corrections are included. We will show that at small angles there is a non-analytic term of the form $|\theta_{12}|^p$ with a power, p , that we will compute. This power is intimately related to the singularities in the first line of (4.5) as a function of $k_1.k_2$.

Let us first understand how the singularities in the flat space answer (4.5) arise. These singularities are at $\alpha' k_1.k_2 = -1 - 2n$. We can rewrite this condition as

$$t \equiv -(k_1 + k_2)^2 = \frac{2 + 4n}{\alpha'}, \quad n = 0, 1, 2, \dots \quad (4.18)$$

Similar looking singularities are a well known feature of string scattering amplitudes and they arise when an invariant, such as t , is equal to the mass of a string state. In that case we can view them as arising from the production of an on-shell closed string state. In our case, however, there are no states in the closed string spectrum with masses given by (4.18). Thus we seem to have a puzzle. We will argue that we indeed have certain string states, but of a non-local kind.

Let us first understand the worldsheet origin of these singularities. For concreteness, let us focus on the first singularity at $\alpha' k_1.k_2 = -1$. We see that at this point the integral in (4.5) diverges at $\sigma = 0$ like $\int \frac{d\sigma}{\sigma}$. At $\sigma \sim 0$ the two closed string vertex operators of the

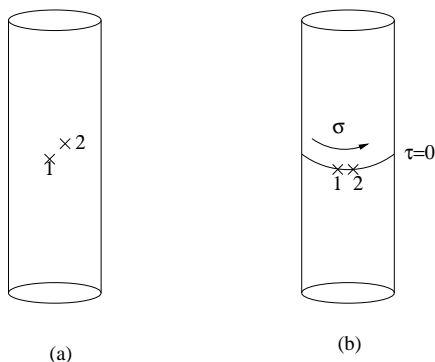


Figure 7: (a) The poles of the ordinary closed string amplitude arise from the region where the two vertex operators are close to each other but are integrated over both τ and σ . (b) Worksheet OPE for the problem we are considering where we have wavefunctions localized in x^+ . In light cone gauge this results in operators localized at $\tau = 0$. Thus we get singularities from the region of the integral where $\sigma_{12} \rightarrow 0$.

external gravitons come close together, see 7 b). This looks similar to the ordinary OPE region of a closed string worldsheet which produces the usual closed string state poles. The crucial difference is that in our case the integral runs only over the sigma direction (see 7 b), while in the ordinary case it runs over τ and σ (see 7 a). For this reason the position of the poles has been shifted compared to the ordinary closed string poles. Schematically we have

$$\begin{aligned}
 \text{Usual Case :} & \quad \int dz^2 |z|^{\alpha' k_1 \cdot k_2} \sim \frac{1}{\alpha' k_1 \cdot k_2 + 2} \\
 \text{Our Case :} & \quad \int d\sigma |\sigma|^{\alpha' k_1 \cdot k_2} \sim \frac{1}{\alpha' k_1 \cdot k_2 + 1}
 \end{aligned} \tag{4.19}$$

We have now understood how the singularities arise from the worldsheet computation. Before moving on, let us clarify further a confusing aspect of these singularities. All we are doing is to scatter four string states: the two gravitons, the state we are measuring and its complex conjugate. So, why are the singularities different than singularities of the ordinary closed string scattering amplitude? What happens is that we are choosing very peculiar wavefunctions for the two gravitons. These wavefunctions contain $\delta(y^+)$ factors which implicitly carry an infinite amount of momentum. More precisely, in order to go from the usual momentum space result to our expression for $\delta(y^+)$ wavefunctions we should integrate the momentum space result over k_{1+} and k_{2+} . The four point amplitude is characterized by $t = -(k_1 + k_2)^2$ and $s = -(p + k_1)^2$, where p is the momentum of the incoming closed string state with non-zero p_- . Since $k_{i-} = 0$, we have that t is independent of k_{i+} and it continues to be given by the transverse components of k_i , $t = -(\vec{k}_1 + \vec{k}_2)^2$. On the other hand s contains a contribution of the form $s = 4p_- k_{1+} + \dots$. For the polarizations of the

gravitons that we are choosing here the amplitude has the form [70]¹⁷

$$\mathcal{A}_4 = p_-^4 \left(\frac{1}{s} + \frac{1}{u} \right) \frac{\Gamma(1 - \frac{\alpha's}{4})\Gamma(1 - \frac{\alpha't}{4})\Gamma(1 - \frac{\alpha'u}{4})}{\Gamma(1 + \frac{\alpha's}{4})\Gamma(1 + \frac{\alpha't}{4})\Gamma(1 + \frac{\alpha'u}{4})}, \quad u = -s - t \quad (4.20)$$

In the gravity limit we recover the results that we expect from the diagram in 5 (a) and a crossed version. We can take p_- to be fixed. Then the integral over k_{1+} translates into an integral over s .¹⁸ At large s , the four point amplitude is controlled by Regge behavior. The amplitude goes as

$$\mathcal{A}_4 \sim s^{-2 + \frac{\alpha't}{2}} \quad (4.21)$$

So the integral over s converges at large values of s for small t .¹⁹ As we increase t , the integral over s first diverges when the amplitude goes like $1/s$, $\mathcal{A}_4 \sim 1/s$. This condition is precisely the $n = 0$ case in (4.18). We get the higher order ones by a similar reasoning by expanding the amplitude to higher orders in the $1/s$ expansion. One minor subtlety is that only even powers of $1/s$ give rise to $1/s$ terms in the full amplitude, after we adjust t appropriately.²⁰ Odd powers of $1/s$ would lead to extra singularities beyond those given by (4.18), see more on this below. Thus, we see that the poles (4.18) are associated to the high energy behavior of the string amplitude. This is to be expected since by probing the string at $y^+ = 0$ we are taking a snapshot of the string state and this requires high energy scattering. The fact that the amplitudes we are computing are finite is related to the fact that the high energy scattering displays Regge behavior. In conclusion, the result we obtained in lightcone gauge is perfectly consistent with the usual structure of the Shapiro-Virasoro amplitude.

A related remark that we can make at this point is the following. Let us go back to the case where we consider a neutral falling state probed by two closed string gauge bosons. Going to ten dimensions we can view the gauge bosons as Kaluza Klein gravitons. In that case the flat space amplitude is very similar to (4.20) except that we now have

$$\mathcal{A}_g = p_-^2 \frac{\Gamma(1 - \frac{\alpha's}{4})\Gamma(1 - \frac{\alpha't}{4})\Gamma(1 - \frac{\alpha'u}{4})}{\Gamma(1 + \frac{\alpha's}{4})\Gamma(1 + \frac{\alpha't}{4})\Gamma(1 + \frac{\alpha'u}{4})} \quad (4.22)$$

If we take the small momentum limit of this amplitude we get a constant. If we integrate this constant with respect to s , in order to go to δ function wavefunctions, then we get an infinity. This is the infinity that we mentioned above as coming from the field theory diagram in 5 (b). Of course the full string amplitude is not constant. Thus, once we go to string theory we should integrate the full string amplitude (4.20), which goes as

¹⁷We take the momenta to be nonvanishing only in the first five dimensions. The falling string state is taken to be a graviton which has indices in the remaining five dimensions. Of course the two probe gravitons have indices in the $++$ directions.

¹⁸The integral over k_{2+} simply gets rid of the momentum conservation delta function in the k_+ direction.

¹⁹There are poles along the real s axis. As usual, we give these poles a small positive or negative imaginary part so that we are analyzing the amplitude in the physical sheet. Thus, the poles along the real s axis do not lead to divergences in the amplitude.

²⁰In other words, $A_4 \sim s^{-2 + \alpha't/2} \sum_{n=0}^{\infty} c_n(\alpha't)/s^n$, but $c_{2k+1}(\alpha't = 4k + 4) = 0$.

$\mathcal{A}_g \sim s^{-1+\frac{\alpha' t}{2}}$ and converges if t is negative. We can define it by analytic continuation for other values of t . Thus, we see that for this particular case, taking the low energy limit of the amplitude first and then doing the s integral gives a very different answer than doing first the s integral and then the low energy limit, which gives (4.13). In the case of energy correlations, if we first take the low energy limit of (4.21) and then we do the integral we get the same answer as doing first the integral and then the low energy limit. In this case this happens because the contribution is coming mainly from the $s \sim 0$ and $u \sim 0$ region.

It is useful to perform explicitly the worldsheet operator product expansion of the two graviton vertex operators, see figure 7 (b). We obtain²¹

$$p_- e^{ik_1 \cdot y(\tau=0, \sigma)} p_- e^{ik_2 \cdot y(0,0)} \sim p_-^2 |\sigma|^{\alpha' k_1 \cdot k_2} [e^{i(k_1+k_2)y(0,0)} + \dots] \quad (4.23)$$

The pole arises when the power of σ is precisely $1/\sigma$. This gives rise to the $n = 0$ case in (4.18). The operator that appears at this point has the form

$$p_-^2 e^{ik \cdot y}, \quad m^2 = -k^2 = \frac{2}{\alpha'} \quad (4.24)$$

This is the operator that appears on the worldsheet in light cone gauge. One is tempted to write an operator in conformal gauge that would reduce to (4.24) in light-cone gauge. Due to its peculiar p_- dependence, we are forced to write an expression of the form²²

$$(\partial_\alpha y^+ \partial_\alpha y^+)^{\frac{3}{2}} \delta(y^+) e^{ik \cdot y} \quad (4.25)$$

with k obeying the condition (4.24). This operator is formally a Virasoro primary but is not a proper local operator on the string worldsheet. Similar operators were shown to control Regge physics in [71]. Of course, our regime is closely connected with Regge physics so it is not a surprise that similar operators appear. The operator (4.25), without the delta function, has spin $j = 3$ in the y^+, y^- plane, as we had in the field theory discussion. This is related to the factor of p_-^2 that appears in (4.23). Notice that the complete operator (4.25), including the delta function, has total spin 2. It corresponds to the field theory operator \mathcal{U}_{j-1} with $j = 3$.

The operator (4.24) is the leading contribution in the worldsheet OPE (4.23). As we expand the exponentials at higher orders we pick up new operators which contain derivatives with respect to the transverse directions. Some of these operators can have transverse spin. These strings states have higher masses. They all have spin $j = 3$ in the y^\pm directions. Notice, however, that terms that come with odd powers of σ in the ... in (4.23) vanish upon integration over σ . The reason is a Z_2 symmetry from interchanging $\sigma \rightarrow -\sigma$. This is completely analogous to the fact that terms that are not symmetric under the interchange of $z \rightarrow \bar{z}$ in the usual case where we integrate over the whole complex plane vanish upon integration. This is nothing else than the level matching condition. These

²¹In the following expression it is convenient to redefine the range of σ to $[-\pi, \pi]$, such that insertions close to the operator at zero are given by small $|\sigma|$.

²²Notice that the extra power of p_- appears to compensate the one appearing from the delta function $\delta(y^+) \sim \frac{\delta(\tau)}{p_-}$.

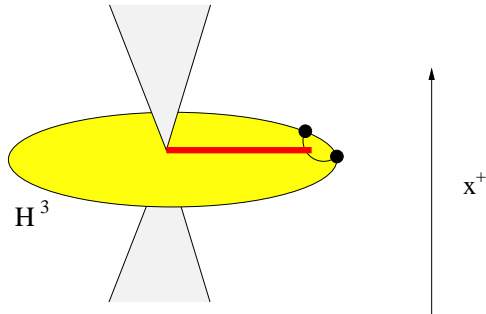


Figure 8: Small angle expansion of the energy correlation function. The expansion is dominated by the propagation of a spin three non-local string state, denoted here by a thick red line.

terms are the same as the ones discussed above in connection with the singularities for odd powers of $\frac{1}{s}$. Now we understand why these poles are absent from (4.5).

In the end, the singularities arise from a fairly ordinary worldsheet operator product expansion in light cone gauge. On the other hand, we cannot associate the corresponding worldsheet operator to an ordinary closed string state. This is related to the fact that the operator product expansion of two energy flux operators in the field theory lead to non-local operators in the y^- direction. In the field theory we were not too disturbed by the appearance of operators that are non-local in the y^- direction, so we should also not be surprised that in string theory we also get string states that are non-local in the y^- direction. These string states are localized at $y^+ = 0$, carry zero p_- and are local in the transverse directions.

The final conclusion of this discussion is that we should interpret the singularities in (4.5) as arising from the propagation, in the transverse space, of non-local string states created by the operators like (4.24) or (4.25).

We will now argue that the short distance singularity of the energy flux two point function is governed by the operator associated to the first singularity in (4.5). Since we are working in the regime of large but fixed λ we might imagine that we could always expand the two point function as in the second line of (4.5). This is not correct if the angle is very small. In that case the relevant relative momenta are of order $t \sim \frac{1}{|\theta_{12}|^2} \frac{1}{R_{\text{AdS}}^2}$. Thus, at angles of order $\theta \sim \lambda^{-1/4}$ we cannot use the approximations leading to (4.9). However, we can use the interpretation given above to the poles in t to write the flat space result in the first line of (4.5) as a sum over contributions of poles. Then each pole corresponds to the contribution of a physical (but non-local) string state that is localized in the y^+ direction but propagating in the transverse directions. We generalize this result to AdS by replacing the transverse space by H^3 . Now the non-local string states propagate on the H_3 subspace of AdS_5 . These states propagate from the center of H_3 , where the string state created by the localized operator insertion is concentrated, to a region near the H_3 boundary, near the insertion of the two energy flux operators, see 8. At large distances from the H_3 center we expect that the wavefunction of the non-local string state goes as $1/|\vec{W}|^\Delta$, $|\vec{W}| \gg 1$, with

$$\Delta \sim mR_{\text{AdS}} \sim \sqrt{2}\lambda^{1/4} + \dots \quad (4.26)$$

where m is given, to a good approximation, by the mass of the flat space state computed in (4.24). Incidentally, we can calculate the conformal weight Δ of other (generally non local) operators with arbitrary spin in the same manner. They correspond to the string states

$$(\partial_\alpha y^+ \partial_\alpha y^+)^{\frac{j}{2}} \delta(y^+) e^{ik \cdot y} \quad (4.27)$$

The mass of these states is given in flat space by $m^2 = -k^2 = \frac{2}{\alpha'}(j-2)$. Therefore

$$\Delta(j) \sim \sqrt{2} \sqrt{j-2} \lambda^{1/4} + \dots \quad (4.28)$$

This formula is expected to be a good approximation only for $j \ll \lambda^{1/2}$, since it was derived assuming the flat space approximation. For very large values of j we get a logarithmic behavior in j , see [77]. Of course this is simply the analytic continuation of the leading Regge trajectory.

The dots in (4.26) and (4.28) denote terms independent of λ as well as higher order corrections. We then need to compute the overlap of a wavefunction which decays like $1/|\vec{W}|^\Delta$ for large $|\vec{W}|$ with the wavefunctions of the two gravitons associated to the energy flux insertions. We find a behavior

$$\langle \mathcal{E}(\theta_1) \mathcal{E}(\theta_2) \dots \rangle \sim \theta_{12}^{\Delta-6} \langle \mathcal{U}_{3-1}(\theta_2) \dots \rangle \quad (4.29)$$

where \mathcal{U}_{3-1} is related to the lightest spin 3 non-local operator, with zero p_- , which at strong coupling has a large dimension (4.26). In string theory this expectation value is computed by inserting the operator (4.24).

In conclusion, the structure of the OPE is precisely what we expected from general principles in any conformal field theory. At weak coupling the operator product expansion is dominated by operators of twist slightly bigger than two. This leads to correlation functions that are highly localized along certain jet directions. For any value of the coupling the operator, or string state, that dominates has zero p_- and spin $j=3$. At strong coupling, the operator acquires a large twist given in (4.26). The fact that operators with spin $j > 2$ have large dimensions at strong coupling is seen to be intimately related with the fact that the energy distribution is uniform. Of course, this fact is also connected with the validity of the gravity approximation in the bulk.

5. Summary, conclusions and open problems

Let us summarize some of our results.

We studied energy correlation functions in conformal field theories. Energy correlation functions are an infrared finite quantity that is useful for characterizing the states produced by localized operator insertions in a field theory [7, 38, 13].

They can be computed for all values of the coupling since they involve the stress tensor operator [9] and make no reference to a partonic description. This is more manifest at strong coupling where the partons are difficult to see in the gravity or string description.

After a conformal transformation these energy correlation functions amount to measuring the state along a null surface. More precisely, each ‘‘calorimeter’’ insertion corresponds to an integral of the stress tensor along a lightlike line, $\int dy^- T_{--}$, (2.6).

We have argued that the small angle behavior of the energy correlation functions is controlled by an operator product expansion which features non-local light-ray operators of definite spin. When two calorimeters come close to each other we have a spin three operator $\langle \mathcal{E}(\theta_1)\mathcal{E}(\theta_2)\cdots \rangle \sim |\theta_{12}|^{\tau_3-4}\langle \mathcal{U}_{3-1}(\theta_2)\cdots \rangle$. These operators can be discussed for any coupling. We recalled the weak coupling expression for the twist [31] (2.22), and we also computed the twist at strong coupling $\tau \sim \sqrt{2}\lambda^{1/4}$ (4.26), after having identified the string states that are dual to the operator \mathcal{U}_{3-1} . These are not ordinary closed string states. They are peculiar string states localized along $x^+ = 0$ that have non-local vertex operators on the covariant worldsheet but do have a local description on the worldsheet in light-cone gauge. Closely related string states appear in the Regge limit [71]. Despite their unfamiliar features they control the short distance singularities of energy correlation functions.

The light-ray operators that appear in the small angle behavior of the correlator are related to the ones that control the moments of the parton distribution functions. In fact, one can write a precise relation between the energy correlation functions on a special state and a particular moment of the functions that govern the deep inelastic scattering amplitude (2.49).

We have seen that energy flux one point functions in states created by currents or stress tensor insertions have an “antenna” pattern²³ which is determined by the three point functions in the conformal field theory (2.33) (2.37). In the gravity description this pattern is spherical but as we include higher order corrections to the gravity action we start seeing deviations from the spherical pattern (3.20) (3.25). These deviations are sensitive to the spin of the operator that created the excitation in the conformal field theory. In the particular case of $\mathcal{N} = 4$ super Yang Mills, the energy one point function is spherical for all values of the coupling. In more general $\mathcal{N} = 1$ superconformal theories we find that the antenna pattern, (2.36) (2.39), depends on the parameters a and c that characterize the three point functions in the current/stress tensor multiplet [46, 45]. These results are exact expressions, valid for any coupling. They depend only on the two anomaly coefficients a and c defined in [46]. Demanding that the energy that calorimeters measure is positive we get a constraint on a and c , $|a - c| \leq c/2$, which is saturated for free field theories (of course, $c > 0$).

We gave a general prescription for computing the energy correlation functions on the gravity side. The operator insertion in the field theory produces a string state that falls into the AdS horizon. Energy correlation functions depend on the wavefunction of this string state at the AdS horizon. The falling string is probed by particular shock waves associated to the insertion of each calorimeter. This can be computed in a simple way by choosing a coordinate system in AdS_5 that is non-singular at the horizon. We can view the computation of the energy correlation functions as taking a snapshot of the falling string state as it crosses the horizon. In the gravity approximation the result depends only on the momentum distribution of the initial state and it is independent of the spin or any other property of the string state we consider. If the state carries a purely timelike momentum $q^\mu = (q^0, \vec{0})$, then the energy distribution on the detector is perfectly spherical with no

²³Remember that individual events do not present this pattern. This refers to the one point functions which consist of averages over events.

fluctuations. As we include stringy corrections we find small fluctuations that are inversely proportional to the square of the radius of AdS in string units (or $1/\sqrt{\lambda}$) (4.9) (4.10). These fluctuations are small but they are not gaussian (4.12). Since the shock waves we are considering are infinitely localized one might worry that this leads to divergences. In fact, they would lead to divergent answers in a field theory context (at least in some cases). The Regge behavior of string amplitudes at large energies ensures that the results we obtain are finite.

It should be fairly straightforward to generalize this discussion to other dimensions. The discussion in 2+1 dimensions might have some condensed matter applications, similar to [72].

It would also be interesting to understand finite N corrections.

There has been a great deal of progress in computing perturbative scattering amplitudes in $\mathcal{N} = 4$ super Yang Mills, see [73] for example. From these scattering amplitudes one can compute the energy correlation functions. On the other hand, since the energy correlation functions are already infrared finite, it would be nice to see if any of the methods developed to compute amplitudes could be extended to compute the energy correlation functions directly, without having to compute the amplitudes first.

It would also be interesting to consider the “hadronization” corrections for a non-conformal theory. In particular one could imagine a non-conformal theory with a gravity dual. In a confining theory with a gravity dual we expect that these corrections will be large in the large N limit because the strings cannot break. One could also try to understand situations where the theory becomes free in the IR, such as 4 + 1 dimensional super Yang-Mills, or dimensionally regularized $\mathcal{N} = 4$ super Yang Mills.

It would also be interesting to generalize this discussion to more complicated initial states such as the one resulting from the collision of two closed string modes in the bulk. This would be analogous to pp collisions.

Finally, this discussion might have some implication for black holes, since energy correlations are a way of measuring the final state of Hawking radiation and its non-thermal properties.

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A. Positivity of $\int dy^- T_{--}$

Let us consider first free field theories. The classical expression for the stress tensor for the Maxwell field, $T_{--} \sim \sum_{i=1,2} F_{-i} F_{-i}$, is explicitly positive since it is a sum of squares. On the other hand, the quantum expectation value of T_{--} can be negative. Let us recall why this happens. Formally, we also have the sum of squares of hermitian operators, so that we would also expect a positive answer. However, when we normal order we subtract and

infinite constant. Then the normal ordered expression is not a sum of squares of hermitian operators. In fact, we have schematically $T_{--} \sim (a^\dagger)^2 + a^\dagger a + a^2$ where we have separated the operator into terms with different numbers of creation and annihilation operators. By considering a state of the rough form $|\Psi\rangle = |0\rangle + \epsilon a_1^\dagger a_2^\dagger |0\rangle$, and using that the vacuum expectation value of T_{--} is zero we find that $\langle\Psi|T_{--}|\Psi\rangle \sim \text{Re}[c\epsilon] + o(\epsilon^2)$ where c is some number. By taking ϵ to be a small complex number we see that we can make T_{--} negative at a point [74].

Let us now consider the integrated expression $\mathcal{E} = \int dy^- F_{-i} F_{-i}$. This expression has the schematic form

$$\mathcal{E} \sim \int_0^\infty dp^+ p^+ (a_{p^+}(\vec{y}))^\dagger a_{p^+}(\vec{y}) \quad (\text{A.1})$$

we thus see that we have the integral of products of operators and their adjoints. This is an explicitly positive operator. We have used the variable $p^+ \sim -2p_-$ which is positive. Notice that terms with two a^\dagger or two a operators have disappeared from (A.1) due to the following argument. The integral over y^- enforces that the total p^+ should be zero. However, creation operators can only increase p^+ , thus we do not obey the $p^+ = 0$ constraint with only creation operators. Further discussion on the null energy condition for free fields can be found in [17].

Let us consider now an interacting field theory. If we choose the gauge $A_- = 0$, then the stress tensor T_{--} continues to be quadratic in the fields and the above argument would hold. Of course, this argument is not too convincing since we might be ignoring renormalization subtleties or problems with the gauge choice. It would be nice to find a more general and solid argument for an interacting field theory.

B. Energy distributions in gravity for generic states

B.1 Energy distributions for general states

Here we show how to go between a discussion of n point functions and the computation of probabilities for seeing various energy distributions on the detector. Sometimes one might be interested in computing the probability functional $\rho[\mathcal{E}(\theta)]$ for measuring a particular pattern of energy deposition on the calorimeters. When one computes jet amplitudes one is computing probabilities of this kind, where one integrates over certain regions, such as the low energy region between two jets, etc.

If we are given ρ we can compute the n point functions, as $\langle\mathcal{E}(\theta_1)\cdots\mathcal{E}(\theta_n)\rangle = \int \mathcal{D}\mathcal{E} \rho[\mathcal{E}] \mathcal{E}(\theta_1)\cdots\mathcal{E}(\theta_n)$.

Formally one can also go in the other direction by computing the generating functional for energy correlation functions, $\langle e^{i \int d^2\theta \lambda(\theta) \mathcal{E}(\theta)} \rangle$. This expression is a functional of $\lambda(\theta)$ and its expansion in powers of λ gives us the n point functions. Then ρ is given by

$$\rho[\mathcal{E}'] = \int \mathcal{D}\lambda e^{-i \int d^2\theta \lambda(\theta) \mathcal{E}'(\theta)} \langle e^{i \int d^2\theta \lambda(\theta) \mathcal{E}(\theta)} \rangle \quad (\text{B.1})$$

Just in order to see how this works, let us start with the n point functions given in (3.38). We can easily compute the expression

$$\langle e^{i \int d^2\theta \lambda(\theta) \mathcal{E}(\theta)} \rangle = \int d^4q \rho(q) e^{i \int d^2\theta \lambda(\theta) \mathcal{E}_{q^\mu}(\theta)} \quad (\text{B.2})$$

where $\mathcal{E}_{q^\mu}(\theta)$ is the function in (3.36) and ρ is defined in (3.38). After doing the functional integral in (B.1) we get

$$\rho[\mathcal{E}] = \int d^4q \rho(q) \prod_{\theta} \delta[\mathcal{E}(\theta) - \mathcal{E}_q(\theta)] \quad (\text{B.3})$$

We see that we have a continuum of δ functions, one for each angle. But we are integrating only over four variables. Thus, once we fix the energy at four points, the energy at all other points is also fixed.

The general, formal, string theory expression for the energy correlators has a similar form, except that we have to integrate over the infinite number of variables specifying the string wavefunction. At finite N we would also have many string states.

B.2 Bulk wavefunction for a localized state

Here we start with the wavefunction $\phi_0(x) \sim e^{-iq^0 t} e^{-\frac{(t^2 + \vec{x}^2)}{\sigma^2}}$ that we mentioned in (2.28). Its Fourier transform is

$$\tilde{\phi}_0(p) = \int d^4x e^{-ipx} \phi_0(x) = \int d^4x e^{ip^0 t - i\vec{p}\vec{x}} e^{-iq^0 t - \frac{(t^2 + \vec{x}^2)}{\sigma^2}} \sim \sigma^4 e^{-\frac{\sigma^2}{4} [(p^0 - q^0)^2 + (\vec{p})^2]} \quad (\text{B.4})$$

The bulk wavefunction then has the form

$$\phi(W^+ = 0, W^-, W^\mu) \sim (q^0)^\Delta \int_0^\infty d\tilde{\lambda} \tilde{\lambda}^{\Delta-1} e^{i\tilde{\lambda} W^- q^0/2} e^{-\frac{(\sigma q^0)^2}{4} [(\tilde{\lambda} W^0 - 1)^2 + |\tilde{\lambda} \vec{W}|^2]} \quad (\text{B.5})$$

We are considering the case that $q^0 \sigma \gg 1$. We then see that as soon as $|\vec{W}| \gg 1/(\sigma q^0)$ the answer is exponentially suppressed. In the region with large $|\vec{W}| \sim W^0 \gg 1$ we can do the integral (B.5) by saddle point approximation and we find

$$\phi(W) \sim (q^0)^\Delta (W^0)^{-\Delta} \frac{1}{(\sigma q^0)} e^{iq^0 W^- / (2W^0)} e^{-\frac{(\sigma q^0)^2}{8}} \quad (\text{B.6})$$

We then insert this in (3.34) to find an expression of the approximate form

$$\langle \mathcal{E}(\vec{n}'_1) \mathcal{E}(\vec{n}'_2) \rangle \sim q_0^2 e^{-\frac{(\sigma q^0)^2}{4}} \int d\Sigma_3 \frac{1}{(W^0)^{\Delta+2}} \frac{1}{(W^0 - \vec{W} \cdot \vec{n}'_1)^3} \frac{1}{(W^0 - \vec{W} \cdot \vec{n}'_2)^3} \quad (\text{B.7})$$

where we used that N^2 in (3.34) is not exponentially small, and in (B.7) we have kept only the leading exponential behavior in σq^0 . We have also approximated the integrand in the large W^0 region which we expect to dominate for the singular small angle behavior of the two point function. Finally we find the singular small angle behavior

$$\langle \mathcal{E}(\vec{n}'_1) \mathcal{E}(\vec{n}'_2) \rangle \sim |\theta|^{2\Delta-4} e^{-(\sigma q^0)^2/4} \quad (\text{B.8})$$

This is precisely the power we expect for the double trace contribution as was discussed in [75]. This term is exponentially suppressed when we consider a state with definite momentum. Thus, the term that gave the largest contribution in the deep inelastic scattering analysis in [75] does not contribute to energy correlators when we consider states created with definite momentum. They do contribute if the state does not have definite momentum in x -space. In fact, we saw that a state with definite momentum in y space is directly connected with the deep inelastic scattering amplitudes (2.49). Such a state does not have definite momentum in x -space and will not have the exponential suppression that we get in (B.8).

We should emphasize that the contribution to (B.8) is coming from the region where the particle is crossing the horizon at a position that is close to where the calorimeters are inserted.

C. Computation of the energy and charge one point function in terms of the three point functions of the CFT

We denote by \mathcal{O} any operator, which could be a scalar operator \mathcal{S} or a vector $\epsilon.j$ or a tensor $\epsilon_{ij}T_{ij}$.

Let us start by recalling some formulas for two point functions

$$\begin{aligned}\langle 0|S(t,x)S(0,0)|0\rangle &= \frac{1}{[-(t-i\epsilon)^2+|\vec{x}|^2]^\Delta} \\ \langle 0|T(S(t,x)S(0,0))|0\rangle &= \frac{1}{[-t^2+|\vec{x}|^2+i\epsilon]^\Delta}\end{aligned}\tag{C.1}$$

where the first is not time ordered and the second is time ordered. Of course the same prescription works for vector or tensor operators. The operator insertion with a definite timelike momentum $q^\mu = (q^0, \vec{0})$ can be written as

$$\mathcal{O}_q|0\rangle = \int dt e^{-iq^0 t} \mathcal{O}(t)|0\rangle\tag{C.2}$$

and it creates a state with energy $E = q^0 > 0$. The fourier transform of the two point function is

$$\int d^4x e^{-iq.x} \frac{1}{[-(t-i\epsilon)^2+|x|^2]^\Delta} = c(\Delta)\theta(q^0)(-q^2)^{\Delta-2}, \quad c(\Delta) = \frac{(2\pi)^3(\Delta-1)}{4^{\Delta-1}\Gamma(\Delta)^2}\tag{C.3}$$

This is the norm of the state that (C.2) creates. This will also give us the total production cross section if the operator \mathcal{O} couples to the standard model. As remarked in [76] the positive norm condition implies $\Delta \geq 1$.

We are interested in starting from the ordinary expressions for the correlation functions in position space and extracting the limit that corresponds to the energy or charge correlators. In doing so, it is important to order the operators appropriately. For the non-time-ordered three point function the correct prescription is

$$\begin{aligned}\langle 0|\mathcal{S}(x_2)\mathcal{S}(x_1)\mathcal{S}(x_3)|0\rangle &= \\ &= \frac{1}{\{[-(t_{23}-i\epsilon)^2+(\vec{x}_{23})^2][-(t_{13}-i\epsilon)^2+(\vec{x}_{13})^2][-(t_{21}-i\epsilon)^2+(\vec{x}_{21})^2]\}^{\Delta/2}}\end{aligned}\tag{C.4}$$

If one considers tensor operators we get similar denominators and we choose the same $i\epsilon$ prescription. This $i\epsilon$ prescription is a simple way to enforce the right ordering of the operators. Another way to say this is that an operator that is to the ‘left’ of another should have a more negative imaginary part in the time direction. When one does perturbation theory, it might be convenient to use time ordering along a Keldysh contour. However, for our purposes this simple prescription suffices.

Let us first show how to extract the energy correlation for a state created from a scalar operator with fixed momentum, or at least fairly well defined momentum, as in (2.28). In this case, we know that the answer is independent of the angles and that the overall coefficient is determined by energy conservation. Nevertheless it is instructive to discuss this case in detail since the computation is the simplest and one can apply a similar method for other cases. Our method is not too elegant and there is probably a more direct and elegant method than the one we applied here.

We extract the energy correlation by directly performing the limit and the integral in (1.1). We use translation invariance to fix the position of the first operator at $x_3 = 0$. For simplicity we place the detector along the direction z , so that $x_1 = (t, 0, 0, r)$. We will take the limit $r \rightarrow \infty$. If t is generic, then the three point function will decay as $1/r^8$ since it would be determined by the dimension of the stress tensor and the operator product expansion. This would be decaying too rapidly in order to give a finite large r limit. However, there is a larger contribution from the region $t \sim r$, the region on the light-cone of the inserted operators. This is the region that will contribute. Of course, this is precisely what we would expect in a theory of massless particles. It is convenient to define coordinates $x^\pm = t \pm r$. We will find that the region with finite x^- will contribute. In addition, only the T_{--} component of the stress tensor can contribute. The integral over t can be traded for an integral over x^- . So we first take the $r \rightarrow \infty$ limit. We then do the integral over x^- and at the end we do the integral over x_2 .

Let us see what each of these steps gives us. In order to follow this appendix the reader would need to have a copy of the paper by Osborn and Petkos [43], since we will make frequent reference to it. We start with the correlation function for

$$\langle 0 | \mathcal{S}(x_2) T_{--}(x_1) \mathcal{S}(x_3) | 0 \rangle \sim \frac{1}{x_{23}^{2\Delta-2} x_{12}^2 x_{13}^2} \left(\frac{x_{12}^+}{x_{12}^2} - \frac{x_{13}^+}{x_{13}^2} \right)^2 \quad (\text{C.5})$$

from equation (3.1) of [43]. We are not going to keep track of overall numerical coefficients. After multiplying by r^2 and taking the $r \rightarrow \infty$ limit we get

$$\lim_{r \rightarrow \infty} r^2 \langle 0 | \mathcal{S}(x_2) T_{--}(x_1) \mathcal{S}(0) | 0 \rangle \sim \frac{(x_2^-)^2}{(x_2^2)^{\Delta-1}} \frac{1}{(x^- - i\epsilon)^3 (x^- + i\epsilon - x_2^-)^3} \quad (\text{C.6})$$

We now perform the integral over x^- . Note that we can close the contour on either the upper or lower x^- plane and pick one of the two poles in (C.6). We then find

$$\lim_{r \rightarrow \infty} r^2 \langle 0 | \mathcal{S}(x_2) \int dx^- T_{--}(x_1) \mathcal{S}(0) | 0 \rangle \sim \frac{1}{(x_2^2)^{\Delta-1}} \frac{1}{(x_2^- - 2i\epsilon)^3} \quad (\text{C.7})$$

We now integrate over the two transverse x_2 coordinates and use the the wavefunction in (C.2) does not depend on them. We find

$$\begin{aligned} \lim_{r \rightarrow \infty} r^2 \int d^4 x_2 e^{iq^0 t_2} \langle 0 | \mathcal{S}(x_2) \int dx^- T_{--}(x_1) \mathcal{S}(0) | 0 \rangle \\ \sim \int dt_2 dz_2 e^{iq^0 t_2} \frac{1}{[-(t_2 - i\epsilon)^2 + z_2^2]^{\Delta-2}} \frac{1}{(t_2 - z_2 - 2i\epsilon)^3} \sim \theta(q^0) (q^0)^{2\Delta-3} \end{aligned} \quad (\text{C.8})$$

where we have also done the remaining two integrals. When we divide by the two point function (C.3) we get $\langle \mathcal{E} \rangle \sim q^0$. The numerical coefficient can also be computed at each step. Of course, this gives the right answer $\langle \mathcal{E} \rangle = \frac{q^0}{4\pi}$ due to the Ward identity which fixes the coefficient of the three point function (C.5) in terms of the coefficient of the two point function (C.1), see eqns (6.15), (6.20) of [43].

This procedure can be repeated replacing the operator \mathcal{S} by a current $\epsilon.j$. The computations are identical but with more indices and we use a computer. In this case the three point function of a stress tensor and two currents is fixed by conformal invariance, plus the Ward identity, up to one unknown coefficient. Conformal invariance leaves two possible structures and the Ward identity fixes the coefficient of one of them. In this case we find, as expected, that the energy correlation function depends on the angle with respect to the vector ϵ . Here we simply quote the value of the parameter a_2 , introduced in (2.30), in terms of the parameters \hat{e}, \hat{c} defined in (3.13) and (3.14) of [43].²⁴ We find that

$$a_2 = \frac{3(8\hat{e} - \hat{c})}{2(\hat{e} + \hat{c})} \rightarrow 3 \frac{\sum_i (q_i^b)^2 - (q_i^{wf})^2}{\sum_i (q_i^b)^2 + 2(q_i^{wf})^2} \quad (\text{C.9})$$

where we indicated the value for a free theory with bosons and Weyl fermions of charges q_i^b and q_i^{wf} . The combination $(\hat{e} + \hat{c})$ is fixed in terms of the coefficient of the two point function of two currents via 6.26 of [43].

We can do this also for the correlation functions of the form $\langle 0 | \epsilon_{ij}^* T_{ij} \mathcal{E} \epsilon_{ij} T_{ij} | 0 \rangle$. We can then compute the coefficients t_2, t_4 introduced in (2.37)

$$\begin{aligned} t_2 &= \frac{30(13\hat{a} + 4\hat{b} - 3\hat{c})}{14\hat{a} - 2\hat{b} - 5\hat{c}} \rightarrow \frac{15(-4n_v + n_{wf})}{(n_b + 12n_v + 3n_{wf})} \\ t_4 &= -\frac{15(81\hat{a} + 32\hat{b} - 20\hat{c})}{2(14\hat{a} - 2\hat{b} - 5\hat{c})} \rightarrow \frac{15(n_b + 2n_v - 2n_{wf})}{2(n_b + 12n_v + 3n_{wf})} \end{aligned} \quad (\text{C.10})$$

where $\hat{a}, \hat{b}, \hat{c}$ are defined in (3.19)-(3.21) of [43]. We have also indicated the result for n_v, n_b and n_{wf} free vectors, real bosons, and Weyl fermions (one complex dirac fermion would give $n_{wf} = 2$).²⁵ Again, the combination appearing in the denominator is fixed in terms of the stress tensor two point function, see (6.42) of [43].

²⁴Here we added hats to the parameters e and c in [43] so that they are not confused with other parameters in the present paper. Note also that [43] uses some of these letters with multiple meanings through their paper.

²⁵ $\hat{a}, \hat{b}, \hat{c}$ here should not be confused with the \hat{e}, \hat{c} of the previous paragraph. In particular the two \hat{c} are not the same [43].

Two combinations of these three coefficients are related to the values of a and c defined through the conformal anomaly²⁶

$$T^\mu_\mu = \frac{c}{16\pi^2}W^2 - \frac{a}{16\pi^2}E \quad (\text{C.11})$$

where W is the Weyl tensor and $E = R_{\mu\nu\delta\rho}R^{\mu\nu\delta\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the Euler density. c also sets the two point function of the stress tensor. The coefficient a can be expressed in terms of the three parameters in (C.10) as

$$\frac{a}{c} = \frac{(9\hat{a} - 2\hat{b} - 10\hat{c})}{3(14\hat{a} - 2\hat{b} - 5\hat{c})} \rightarrow \frac{2n_b + 124n_v + 11n_{wf}}{6n_b + 72n_v + 18n_{wf}} \quad (\text{C.12})$$

This follows from (8.37) in [43]. The values for a free theories were computed in [47, 48]. From the positivity conditions (2.38) it is possible to get general bounds on this ratio. We find

$$\frac{31}{18} \geq \frac{a}{c} \geq \frac{1}{3} \quad (\text{C.13})$$

where the the lower bound is saturated by a free theory with only scalar bosons and the upper bound by a free theory with only vectors. Notice that this bound holds for any conformal theory while the more restrictive bound (2.40) holds for supersymmetric theories. We can also add that for a $\mathcal{N} = 2$ supersymmetric theory we can find a similar bound by using as limiting cases free theories with only vector supermultiplets ($\frac{a}{c} = \frac{5}{4}$) and free theories with hypermultiplets ($\frac{a}{c} = \frac{1}{2}$). Therefore $\frac{5}{4} \geq \frac{a}{c} \geq \frac{1}{2}$. This agrees with results from [52].

In an $\mathcal{N} = 1$ supersymmetric theory there is a relation between the three parameters $\hat{a}, \hat{b}, \hat{c}$ which is obtained by setting $t_4 = 0$ in (C.10). In this case, the two coefficients in (C.11) specify completely the three point functions of the stress tensor. In a non-supersymmetric theory, we have one more parameter beyond the two in (C.11).

Finally, we can repeat this exercise for the correlation function of three currents to find that the coefficient introduced in (2.43)

$$\tilde{a}_2 = \frac{3}{2} \frac{5\hat{a} - 4\hat{b}}{(\hat{a} + 4\hat{b})} \rightarrow 3 \frac{\sum_i C(r_i)_b - C(r_i)_{wf}}{\sum_i C(r_i)_b + 2C(r_i)_{wf}} \quad (\text{C.14})$$

where \hat{a}, \hat{b} are defined in eq. (3.9) of [43], and are not the same as the ones in the previous paragraphs. Here r_i are the representations of the bosons and Weyl fermions. And $C(r)$ is defined as $tr_r[T^a T^b] = C(r)\delta^{ab}$. Again, the combination $(\hat{a} + 4\hat{b})$ sets the two point function of the current. And \tilde{a}_2 vanishes in a supersymmetric theory since there is only one structure contributing for the supersymmetric case [45] and it vanishes for a free supersymmetric theory.

One can also take similar limits of the parity odd part of the three point function of three currents and one obtains the result in (2.44). Finally, starting from the correlation functions for two stress tensors and a current derived in [54] one can derive the charge distribution function in (2.45). Both of these results relate the corresponding anomaly to a charge asymmetry.

²⁶Do not confuse these a and c with the parameters \hat{a} and \hat{c} of the previous paragraph.

D. Energy one point functions in theories with a gravity dual

In this appendix we present the calculation of energy one point functions for states created by current operators and the stress energy operator.

D.1 One point function of the energy with a current source

We wish to compute the contributions to the energy one point function (2.30) for a state created by a current operator at strong coupling. The *AdS/CFT* dictionary says we need to compute the bulk three point function between two bulk photons and the graviton. The bulk action (3.18) contains two terms. The first term contributes also to the current two point function while the second term in (3.18) does not. Thus, the first term contributes to the part of the energy one point function which is determined by the Ward identity and the second one to the angular dependent term which is not fixed by the Ward identity. In principle, the first term could also contribute to the angularly dependent part, but we have argued, based on the results for $\mathcal{N} = 4$ SYM that it contributes only to the constant part. Thus in order to compute the coefficient a_2 in (2.30) we need to compute the ratio of the contribution of the second term in (3.18) and the contribution of the first term in (3.18).

Since the graviton is localized in W^+ and the photon is localized in the other transverse directions if the state has definite four dimensional momentum, we can approximate the computation as a flat space computation. In this particular case, this approximation will be exact, but that will not be the case when we discuss the three graviton vertex. Thus, we evaluate the vertex expanding the flat space action, but we will insert the AdS wavefunctions for the external states.

In flat space our coordinates are $(x^+, x^-, x^{1,2,3})$. The metric is $ds^2 = -dx^+ dx^- + dx^i dx^i$ (latin indices i and j go from 1 to 3).

We want to collect terms that are of first order in the perturbation h . There are two such terms, one per factor of $g_{\mu\nu}$ in the action. The determinant g does not receive corrections and $\sqrt{g} = \frac{1}{2}$. Our perturbation is of the form $h = h_{++}(x^i, n^i)\delta(x^+)(dx^+)^2$. We made explicit the fact that h_{++} depends on the transverse coordinates and on a unit vector n^i in the transverse space that represents the position of our calorimeter.

Therefore, we want to calculate

$$S_1 = -\frac{1}{4g^2} \int \frac{dx^+ dx^- d^3x}{2} 2h_{++} F^{+i} F^{+j} g_{ij} = -\frac{1}{4g^2} \int dx^+ dx^- d^3x h_{++} F^{+i} F^{+j} g_{ij} \quad (\text{D.1})$$

Notice that the contraction of F 's is restricted to the 3 dimensional transverse space as the metric element g_{--} is zero. We can do the x^+ integral easily as h_{++} is localized in this direction. Also, we will use the fact that the wave function of the photon is localized in the transverse space (3.15). We represent this fact by writing

$$F^{+i} F^{+j} g_{ij} = \alpha(x^+, x^-) \delta^3(x^i) \epsilon^i \epsilon^j g_{ij} = \alpha(x^+, x^-) \delta^3(x^i) \quad (\text{D.2})$$

where ϵ^i represents the (normalized) polarization of the photon. Notice that we choose the polarization in the transverse directions. Therefore $F^{+i} \sim \partial^+ A^i$. Using these facts and

performing the integrals we get

$$S_1 = -\frac{1}{4g^2} \int dx^+ dx^- d^3x h_{++} F^{+i} F^{+j} g_{ij} = -\frac{1}{4g^2} h_{++}(0, n^i) \int dx^- \alpha(0, x^-) \quad (\text{D.3})$$

In fact, h_{++} evaluated at $x^i = 0$ does not depend on n^i , see (3.10) at $W^i = 0$. Therefore, this term does not contribute to the angular dependence of the correlation function.

Let us now look at the other term in (3.18). We first need to compute the Weyl tensor. Let us start with the Riemann tensor. This tensor has terms that go as $\frac{1}{2}\partial^2 g$ and terms that go as $g\Gamma\Gamma$. Since we are in a flat space background only the first type of term contributes. This yields

$$R_{+i+j} = \frac{1}{2} \partial_i \partial_j h_{++} \quad (\text{D.4})$$

All other terms are given by symmetry properties (i.e. $R_{+i+j} = -R_{i++j} = -R_{+ij+} = R_{i+j+}$) or vanish. The Weyl tensor also contain terms of the form $\frac{1}{3}g_{\lambda\nu}R_{\mu\kappa}$. But,

$$R_{\mu\nu} = g^{\lambda\rho} R_{\lambda\mu\rho\nu} \quad \longrightarrow \quad R_{++} = g^{ij} R_{+i+j} = \frac{1}{2} g^{ij} \partial_i \partial_j h_{++} \quad (\text{D.5})$$

We see there is only one non vanishing term that is proportional to the laplacian inside the transverse space. There are also terms proportional to the Ricci scalar inside the Weyl tensor, but we can see that these vanish in our case. The Weyl tensor is, then, given by

$$C_{+i+j} = \frac{1}{2} \left(\partial_i \partial_j - \frac{1}{3} g_{ij} \partial^k \partial_k \right) h_{++} \quad (\text{D.6})$$

The other components either vanish or are given by symmetry properties. There are four possible positions for the two plus signs, so we will have four terms in the second term in (3.18) (we are also using symmetry properties of F^{+i}). That is

$$\begin{aligned} S_2 &= \frac{\alpha}{g^2 M_*^2} \int \frac{dx^+ dx^- d^3x}{2} 4C_{+i+j} F^{+i} F^{+j} \\ &= \frac{\alpha}{g^2 M_*^2} \int dx^+ dx^- d^3x F^{+i} F^{+j} \left(\partial_i \partial_j - \frac{1}{3} g_{ij} \partial^k \partial_k \right) h_{++} \end{aligned} \quad (\text{D.7})$$

Once again we can perform the integrals to obtain

$$S_2 = \frac{\alpha}{g^2 M_*^2} \left(\partial_i \partial_j - \frac{1}{3} g_{ij} \partial^k \partial_k \right) h_{++}(x^i, n^i) \Big|_{x^i=0} \epsilon^i \epsilon^j \int dx^- \alpha(0, x^-) \quad (\text{D.8})$$

We are interested in the quotient between the angularly dependent term (D.8) and the spherically symmetric term (D.3)

$$-\frac{4\alpha}{M_*^2} \frac{\left(\partial_i \partial_j - \frac{1}{3} g_{ij} \partial^k \partial_k \right) h_{++}(x^i, n^i) \Big|_{x^i=0} \epsilon^i \epsilon^j}{h_{++}(0, n^i)} \quad (\text{D.9})$$

We use the explicit form of the perturbation (3.10) and we get the result

$$a_2^{\text{AdS}} = -48 \frac{\alpha}{M_*^2} \quad (\text{D.10})$$

This gives the gravity result for the anisotropic part of the one point function (2.30) of a state produced by a current.

D.2 One point function of the energy with a stress tensor source

Now we want to repeat this calculation for the case where we have the stress tensor as a source. In this case we need to consider 3 graviton interactions. There are 3 operators that contribute to this vertex. A natural parametrization is given by the action (3.24).

We will do first the calculation in a flat space background. As in the computation we did above we will get derivatives acting on the perturbation h . When we go to the AdS background we could get terms involving the background curvature. Such terms are isotropic and will not contribute to the terms that have maximal angular momentum. But they do give contributions to the terms that have smaller values of the angular momentum. The computations we do here give only the leading contribution for t_2 and t_4 in (2.37). We start from the action in (3.24) and we expand each term to cubic order. We focus on terms with highest angular momentum in the transverse dimensions (see We use the fact that we need one of the metric perturbations to be h_{++} while the other two only have purely transverse indices. We find

$$\begin{aligned} R &= -\frac{1}{2}h_{++}h_{(1)}^{ij}(\partial^+)^2h_{(2)}^{ij}, \\ R_{\mu\nu\delta\sigma}R^{\mu\nu\delta\sigma} &= -2\partial_i\partial_jh_{++}h_{(1)}^{ik}(\partial^+)^2h_{(2)}^{jk}, \\ R_{\mu\nu\delta\sigma}R^{\delta\sigma\rho\gamma}R_{\rho\gamma}{}^{\mu\nu} &= -6\partial_i\partial_j\partial_k\partial_\ell h_{++}h_{(1)}^{ij}(\partial^+)^2h_{(2)}^{k\ell} \end{aligned} \quad (\text{D.11})$$

Notice that expanding the determinant of the metric in the action does not contribute to the three point function. If we now use that the wave function is going to be of the form

$$h_{(1)}^{ij}\partial^{+2}h_{(2)}^{k\ell} = \beta(x^+, x^-)\delta^3(\vec{x})\epsilon^{ij}\epsilon^{k\ell} \quad (\text{D.12})$$

we can perform the integrals and calculate the quotients of the contribution to the three point function. After taking the derivatives and evaluating at $\vec{x} = 0$ we obtain the ratios

$$t_2 = 48\frac{\gamma_1}{R_{\text{AdS}}^2M_{pl}^2} \quad (\text{D.13})$$

$$t_4 = 4320\frac{\gamma_2}{R_{\text{AdS}}^4M_{pl}^4} \quad (\text{D.14})$$

Due to the issues we discussed above, there are terms contributing to t_2 which are of first order in $\gamma_2/(R_{\text{AdS}}M_{pl})^4$, coming from the term with six derivatives in the action, which we neglected compared to the contribution of the four derivative terms. These formulas are also valid only to first order in the γ_i .

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